

ALGEBRAIC RELATIONS, TAYLOR COEFFICIENTS OF HYPERLOGARITHMS AND IMAGES BY FROBENIUS - I

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ABSTRACT. This is the first of three papers, which provide an algebraic interpretation - under the concepts of motives and periods - of our results of p -adic analysis on the de Rham pro-unipotent fundamental group of curves $\mathbb{P}^1 - Z$.

In this part I, let us consider the pro-unipotent fundamental group of the spaces $M_{0,n}$. It is known to provide an algebraic theory of multiple zeta values as periods. Four families of universal algebraic relations are the motivic relations - which play a particular role - and the relations of double shuffle, associator and Kashiwara-Vergne.

We take here this theory as a model. We give adaptations of the basics of the theory into similar basics of a theory adapted to a variant ; it describes, on the one hand, certain specific sequences of Taylor coefficients of hyperlogarithms ; and on the other hand, certain specific formal infinite sums of multiple zeta values. The relation between the two is originated in our theorem expressing these Taylor coefficients in terms of their "images by Frobenius" involving infinite sums of p -adic multiple zeta values.

This gives a "prime multiple harmonic sum motive", lying in the weight-adic completion of the algebra of motivic hyperlogarithms, and its various periods, describing conjecturally the algebraic relations among multiple harmonic sums whose upper bound is a power of a prime number.

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1. INTRODUCTION

1.1. The pro-unipotent fundamental group of $M_{0,n}$ and its periods. The notion of pro-unipotent fundamental group can be used as a generic term, which applies to certain algebraic varieties. It arises, primarily, as a collection of fundamental groupoids associated with Tannakian categories of unipotent vector bundles with connection ; this definition being related to the Tannakian point of view on the topological fundamental group.

It has been defined by Deligne in [D] and by Deligne-Goncharov in [DG]. Four versions of the pro-unipotent fundamental group, related to each other by isomorphisms, are defined in [D] : Betti, de Rham, crystalline, and etale l -adic. We will denote them by $\pi_1^{un,B}$, $\pi_1^{un,dR}$, $\pi_1^{un,cris}$, $\pi_1^{un,l}$. An interpretation of $\pi_1^{un,cris}$ in terms of rigid geometry has been provided by Chiarellotto-Le Stum in [CL].

A motivic version $\pi_1^{un,mot}$ is defined in [DG] ; one can apply to it Betti, de Rham, Hodge and l -adic realization functors, mapping it to the versions above. The existence of a crystalline realization has been announced but unpublished by Yamashita.

An interesting aspect of the pro-unipotent fundamental group, which is the theme of this paper, is its application to the arithmetics of periods. Its periods are algebraic (homotopy-invariant) iterated integrals ; the "iteration" of the integral is the counterpart of the unipotence of the bundles that are involved in the definition of π_1^{un} .

Let, for $n \in \mathbb{N}^*$, the variety

$$\begin{aligned} M_{0,n+3} &= \{(x_1, \dots, x_{n+3}) \in (\mathbb{P}^1)^{n+3} \mid \text{for all } i,j, x_i \neq x_j\} / \text{PGL}_2 \\ &\simeq \{(y_1, y_2, \dots, y_n) \in (\mathbb{P}^1 - \{0, 1, \infty\})^n \mid \text{for all } i,j, y_i \neq y_j\} \end{aligned}$$

Consider also the forgetful maps $M_{0,m+3} \rightarrow M_{0,n+3}$, $m > n$, which are fibrations. When $m = n + 1$, the fibers are curves of the form $X_Z = \mathbb{P}^1 - Z$, where Z is a set of $n + 3$ points. When $m = n + 2$, the fibers are surfaces $X_Z^2 - \Delta_Z$, where Δ_Z is the diagonal of X_Z^2 .

For M being either a $M_{0,n+3}$, either a fiber of a forgetful map $M_{0,n+3} \rightarrow M_{0,m+3}$, let us fix a point O , which will serve as an extremity of the paths of integration. It can be either a rational point of M , either a non-zero rational tangent vector \vec{v}_x at a point of $\overline{M} - M$, to \overline{M} , where \overline{M} is the Deligne-Mumford compactification.

The bundle of paths of $\pi_1^{un,dR}(M)$ starting at O , which is a $\pi_1^{un,dR}(M; O)$ -torsor, is equipped with a canonical unipotent connection, called after Knizhnik and Zamolodchikov and denoted by ∇_{KZ} . The associated periods define horizontal sections of ∇_{KZ} when the extremities of the paths vary.

The Betti-de Rham periods of the $\pi_1^{un}(M_{0,n+3})$'s can all be expressed to the two following particular families of periods [Br1]. Their respective definitions are recalled below in §1.2 and §1.3 ; these are :

i) Multiple zeta values, which arise as periods of the π_1^{un} of $M_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$, that is to say the one dimensional case of the spaces $M_{0,n+3}$

ii) Hyperlogarithms, which generalize multiple zeta values, and arise as periods of the π_1^{un} of curves $X_Z = \mathbb{P}^1 - Z$, that is to say the one dimensional case of the fibers of the forgetful maps $M_{0,m+3} \rightarrow M_{0,n+3}$.

1.2. Multiple zeta values and bases of their algebraic theory. Multiple zeta values are the following real numbers with $(s_d, \dots, s_1) \in \coprod_{d \in \mathbb{N}^*} (\mathbb{N}^*)^d$, $s_d \geq 2$:

$$\zeta(s_d, \dots, s_1) = \sum_{0 < n_1 < \dots < n_d} \frac{1}{n_1^{s_1} \dots n_d^{s_d}} \in \mathbb{R}$$

The fact that they are Betti-de Rham periods of the $\pi_1^{un}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)$ is equivalent to their expression as iterated integrals, which is the following. Let $\omega_0 = \frac{dz}{z}$ and $\omega_1 = \frac{dz}{z-1}$, let $n = s_d + \dots + s_1$ (the weight of $\zeta(s_d, \dots, s_1)$) and $(i_n, \dots, i_1) = (\underbrace{0, \dots, 0}_{s_d-1}, 1, \dots, \underbrace{0, \dots, 0}_{s_1-1}, 1)$. We have

$$(1) \quad \zeta(s_d, \dots, s_1) = (-1)^d \int_0^1 \omega_{i_n}(t_n) \int_0^{t_n} \dots \int_0^{t_3} \omega_{i_2}(t_2) \int_0^{t_2} \omega_{i_1}(t_1)$$

Each fiber $\pi_1^{un,dR}(X, x, y)$, and similarly for the other realizations is an affine scheme over \mathbb{Q} . The generating series of multiple zeta values (in a sense that we will make precise in §1.3) is a \mathbb{R} -point

$$\Phi \in \pi_1^{un,dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)(\mathbb{R})$$

Similarly, in the p -adic case, for each k th power of the crystalline Frobenius, with $k \in \mathbb{N}^*$, we have an element :

$$\Phi_{p,-k} \in \pi_1^{un,dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)(\mathbb{Q}_p)$$

expressing the action of Frobenius on a special path ; and p -adic multiple zeta values $\zeta_p(s_d, \dots, s_1) \in \mathbb{Q}_p$ are defined as their coefficients. Finally, we have a motivic version :

$$\Phi_m \in \pi_1^{un,mot}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{1}_0, -\vec{1}_1)$$

which is of particular significance. Motivic multiple zeta values are functions acted upon by the Tannakian group of a category of mixed Tate motives, i.e. they are transcendental numbers with a Galois theory : certain transcendence questions on multiple zeta values can be settled for motivic multiple zeta values by means of this Galois action. This enables to reduce these transcendence questions to the injectivity of the surjective map defined by $\zeta_m(s_d, \dots, s_1) \mapsto \zeta(s_d, \dots, s_1)$ from the \mathbb{Q} -algebra of motivic multiple zeta values to the one of multiple zeta values.

Multiple zeta values, and their p -adic and motivic variants, satisfy three standard families of algebraic relations, whose definition is recalled in §2.3 : these are the double shuffle relations, the associator relations, and the Kashiwara-Vergne relations.

Conjecturally, each of them generate all the algebraic relations over \mathbb{Q} of multiple zeta values.

In the study of algebraic relations among multiple zeta values, most of the attention is usually focused on $M_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ and $M_{0,5}$; this is because the three families

of relations evoked above can entirely be formulated in terms of these two spaces. A variant of this situation will appear in this paper.

1.3. Hyperlogarithms and bases of their algebraic theory. We now consider a curve $X_Z = \mathbb{P}^1 - Z$, with Z a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$, containing $\{0, 1, \infty\}$. Let us denote the elements of $Z - \{0, 1, \infty\}$ by z_1, \dots, z_r ; additionally, $z_0 = 0$ and $z_{r+1} = 1$. Let us also denote, for $i \in \{0, \dots, r+1\}$, by $\omega_{z_i} = \frac{dz}{z-z_i}$.

Let $(z_{i_n}, \dots, z_{i_1})$, a sequence of elements of Z of the form $(\underbrace{0, \dots, 0}_{s_d-1}, z_{i_d}, \dots, \underbrace{0, \dots, 0}_{s_1-1}, z_{i_1})$, with $i_1, \dots, i_d \in \{1, \dots, r\}$. Let, following [G], the hyperlogarithms :

$$\text{Li} \left(\begin{matrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{matrix} \right) (z) = \int_0^z \omega_{z_{i_n}}(t_n) \int_0^{t_n} \dots \int_0^{t_3} \omega_{z_{i_2}}(t_2) \int_0^{t_2} \omega_{z_{i_1}}(t_1)$$

These are multivalued holomorphic functions. They are relative to the choice of the base-point 0, and consist of the case of convergent integrals at 0 only - this is because $z_{i_1} \neq z_0$, but this double restriction is usual and does not lose generality.

The special values of hyperlogarithms at the elements of Z are generalizations of multiple zeta values. Note however that, in this setting, there is a particular relation between hyperlogarithms and their special values following from a simple change of variable :

$$\text{Li} \left(\begin{matrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{matrix} \right) (z) = \text{Li} \left(\begin{matrix} z_{i_d}/z, \dots, z_{i_1}/z \\ s_d, \dots, s_1 \end{matrix} \right) (1)$$

Let us denote the p -adic and motivic analogues of hyperlogarithms by, respectively, $\text{Li}_{p,-k}$ with $k \in \mathbb{N}$, with $k \in \mathbb{N}^*$ still denoting the power of the crystalline Frobenius, and Li_m . Let us also denote generating series - in the sense explained below - at a point $z \in Z$ by Φ_{0z} , and, similarly, $(\Phi_{0z})_{p,-k}$ and $(\Phi_{0z})_m$. They will be defined precisely as points of $\pi_1^{un}(X_Z, \vec{1}_0, z)$ in §2.2.3.

The philosophy of the properties of motivic multiple zeta values evoked above applies more generally to the motivic hyperlogarithms, which are originated in [G]. In the same paper, the double shuffle relations for hyperlogarithms are also written.

There is an important qualitative difference between algebraic relations of hyperlogarithms on $\mathbb{P}^1 - Z$ and on $\mathbb{P}^1 - \{0, 1, \infty\}$. In the general setting of the $\mathbb{P}^1 - Z$'s, the double shuffle relations involve several different curves $\mathbb{P}^1 - Z$ at the same time. Actually, it is when Z is equal to a set $\{0, \infty\} \cup \mu_N$, with $N \in \mathbb{N}^*$ that only the single curve $\mathbb{P}^1 - Z$ is involved.

In this case - the "cyclotomic case" - the structure of the set of solutions to the double shuffle equations has been studied [Ra]; and a variant of the associator equation has been defined and the structure of its set of solutions has been studied [E].

We do not know further analogues of the families of algebraic relations of the previous paragraph. We will obtain here certain further analogues : they will be sometimes analogues in a loose sense ; but they have a particular application to Taylor coefficients.

As in the algebraic study of multiple zeta values, we will be able to restrict ourselves to the varieties of dimension 1 and 2, which are here, namely, the curves X_Z and the surfaces $X_Z^2 - \Delta_Z$, where Δ_Z is the diagonal of X_Z . These are the fibers of the forgetful maps $M_{0,n+4} \rightarrow M_{0,n+3}$ and $M_{0,n+5} \rightarrow M_{0,n+3}$, with $n+3$ being the cardinal of Z .

Notations. Let e_Z be the alphabet $\{e_0, e_{z_1} \dots, e_{z_r}, e_1\}$. The words on e_Z are the usual way to denote the indices of multiple zeta values and hyperlogarithms, via the correspondence : $\left(\begin{smallmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{smallmatrix} \right) \leftrightarrow e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$.

Let the non-commutative algebra of formal power series with variables the letters of e_Z , and coefficients in an algebra R , be denoted by $R\langle\langle e_Z \rangle\rangle = R\langle\langle e_0, e_{z_1} \dots, e_{z_r}, e_1 \rangle\rangle$. The generating series denoted via the letter Φ evoked in §1.2 and this paragraph are elements of such an algebra of power series. Their coefficients are denoted as follows :

$$f = f[\emptyset] + \sum_{\substack{s_d, \dots, s_0 \in \mathbb{N}^* \\ i_d, \dots, i_1 \in Z - \{0, 1, \infty\}}} f[e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} e_0^{s_0-1}] e_0^{s_d-1} e_1 \dots e_0^{s_1-1} e_1 e_0^{s_0-1} \in R\langle\langle e_Z \rangle\rangle$$

For example, with $R = \mathbb{R}$ and $Z = \{0, 1, \infty\}$: $\zeta(s_d, \dots, s_1) = (-1)^d \Phi[e_0^{s_d-1} e_1 \dots e_0^{s_1-1} e_1]$.

1.4. Multiple harmonic sums and Taylor coefficients of hyperlogarithms. We take the same notations with §1.3. Multiple harmonic sums are the following algebraic numbers, which generalize the harmonic series :

$$H_n \left(\begin{smallmatrix} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{smallmatrix} \right) = \sum_{0 < n_1 < \dots < n_r < n} \frac{(z_{i_2}/z_{i_1})^{n_1} \dots (z_{i_{d+1}}/z_{i_d})^{n_d} (1/z_{i_{d+1}})^n}{n_1^{s_1} \dots n_d^{s_d}}$$

They are, at first sight, the finite iterated sums of multiple zeta value type. Precisely, their primary relation to geometry is that they are essentially equivalent, via two slightly different formulas which we will state precisely in §3.2, to the Taylor coefficients of hyperlogarithms at 0. This fact is at the origin of the series expansion of multiple zeta values and their generalizations ; and, on the other hand, of the non-trivial formulas relating multiple harmonic sums with p -adic multiple zeta values and their generalizations, which we will state in §1.4.

Actually, the relationship between multiple harmonic sums and the special values of p -adic hyperlogarithms is particularly intimate ; it is the origin and, indirectly, the main theme of this paper. It is our main motivation and justification to study multiple harmonic sums and make them into a notion of their own, with an emphasis on the case where the upper bound n is a power of a prime number.

What we aim is to view these numbers as periods of the fundamental group. We aim to lift them to motivic periods, and to transfer on them, and on their motivic lifts, the double shuffle, associator and Kashiwara-Vergne relations. We will obtain indeed certain such analogues, sometimes in a loose sense. Applications to p -adic and finite multiple zeta values will be derived in part II, and applications to intrinsic questions on the algebraic relations in part III.

The first object of study of this paper is the collection of the two definitions below. The reason for these precise definitions, including the multiplication by n^{weight} , will be clear in the next parts.

Let us take $s_d, \dots, s_1 \in \mathbb{N}^*$, $z_{i_1}, \dots, z_{i_{d+1}} \in Z - \{0, \infty\}$. Let $w = \begin{pmatrix} z_{i_{d+1}}, \dots, z_1 \\ s_d, \dots, s_1 \end{pmatrix}$.

Definition 1.1. 1) We will call the Taylor coefficients at the origin (of hyperlogarithms) the following sequences of numbers :

$$(\text{Li } \mathcal{T})_O[w] = (n^{s_d + \dots + s_1} H_n(w))_{n \in \mathbb{N}^*} \in \overline{\mathbb{Q}}^{\mathbb{N}^*}$$

2) We will call prime Taylor coefficients at the origin the following sequence :

$$(\text{Li } \mathcal{T})_{O, \text{prime}}[w] = ((p^k)^{s_1 + \dots + s_d} H_{p^k}(w))_{(p,k) \in \{\text{primes}\} \times \mathbb{N}^*} \in \left(\prod_p \overline{\mathbb{Q}_p} \right)^{\mathbb{N}^*}$$

We will obtain algebraic relations that they satisfy by using their relation to Taylor coefficients of hyperlogarithms and algebraic relations between hyperlogarithms.

1.5. Images by Frobenius and analogues. Let us consider a curve $X_Z = \mathbb{P}^1 - Z$, as before, this time over an absolutely non ramified discrete valuation ring of unequal characteristic $(0, p)$, satisfying the assumptions of [J1], [J2]. Let $X_Z^{(p)}$ be the scalar extension by Frobenius of X_Z . Following [D], §11, consider the pull back by Frobenius $F^* \pi_1^{un, dR}(X_Z^{(p)})$ - constructed analytically and not algebraically - of its De Rham fundamental group, and the isomorphism

$$F_* : \pi_1^{un, dR}(X_Z) \xrightarrow{\sim} F^*(\pi_1^{un, dR}(X_Z^{(p)}))$$

characterized by its horizontality, with respect to ∇_{KZ} , on the fundamental torsor of paths starting at a given base point. The horizontality property of F_* is an overconvergent differential equation satisfied by F_* which can be used to express p -adic hyperlogarithms and multiple harmonic sums in terms of each other. The result that we discuss here is the expression of multiple harmonic sums in terms of p -adic hyperlogarithms. For details on what follows, see [J2]. There is a subgroup

$$\Pi_\Sigma \subset \pi_1^{dR, un}(X_Z, \vec{I}_0, \vec{I}_z)(\mathbb{Q}_p(z_1, \dots, z_r))$$

defined by lower bounds of valuations of the coefficients. Let \tilde{W}_z be the set of possible indices of multiple harmonic sums (§1.4) with $z_{d+1} = z$. We have a map of "infinite summation of certain coefficients" :

$$\Sigma : \Pi_\Sigma \rightarrow \mathbb{Q}_p(z_1, \dots, z_r)^{\tilde{W}_z}$$

The "Ihara action" \circ on $\pi_1^{dR, un}(X_Z, \vec{I}_0, \vec{I}_z)(\mathbb{Q}_p)$, which is intimately related, both, to the pro-unipotent part of the motivic Galois action on motivic hyperlogarithms, and to the Frobenius action on $\pi_1^{dR, un}(X_Z, \vec{I}_0, \vec{I}_z)(\mathbb{Q}_p)$, can be pushed-forward by Σ into a "harmonic Ihara action"

$$\circ_h : \Pi_\Sigma \times \Sigma(\Pi_\Sigma) \rightarrow \Sigma(\Pi_\Sigma)$$

Π_Σ contains as an element $(\Phi_{0z})_{p,-k}$ which is the generating series of p -adic hyperlogarithms at z , relative to the k -th power of the crystalline Frobenius. We will denote, for a word $\tilde{w} = \begin{pmatrix} z, z_{i_d} \cdots z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix}$, by $w = e_z e_0^{s_d-1} e_{z_{i_d}} \cdots e_0^{s_1-1} e_{z_{i_1}}$.

Theorem 1.2. Under certain assumptions on the valuations of the z_i 's, we have, for each n ,

$$((p^k n)^{\text{weight}} H_{p^k n}(\tilde{w}))_{\tilde{w} \in \tilde{W}_z} = (n^{\text{weight}} (\Phi_{0z})_{p,-k}) \circ_h (n^{\text{weight}} H_n(w))_{\tilde{w} \in \tilde{W}_z}$$

When $n = 1$, this actually gives an expression of multiple harmonic sums, whose upper bound is a power of a prime number, in terms of infinite p -adic sums of values of p -adic hyperlogarithms :

$$(2) \quad (p^k)^{s_d + \dots + s_1} H_{p^k}(\tilde{w}) = (\Phi_{0z}^{-1} e_z \Phi_{0z})_{p,-k} \left[\frac{1}{1 - e_0} w \right]$$

In terms of the elementary indexation of iterated integrals, this can be rewritten, when $Z = \{0, 1, \infty\}$, as

$$(3) \quad (p^k)^{s_d + \dots + s_1} H_{p^k}(\tilde{w}) = (-1)^d \sum_{d'=0}^d \sum_{l_{d'+1}, \dots, l_d \geq 0} \prod_{i=d'}^d \binom{-s_i}{l_i} \zeta_{p,-k}(s_{d'+1} + l_{d'+1}, \dots, s_d + l_d) \zeta_{p,-k}(s_{d'}, \dots, s_1)$$

and, in the general case, as

$$(4) \quad (p^k)^{s_d + \dots + s_1} H_{p^k}(\tilde{w}) = z_{i_{d+1}}^{-p^k} (-1)^d \sum_{\substack{0 \leq d' \leq d \\ z_{i_{d'}} = z_{i_{d+1}} \\ l_{d'+1}, \dots, l_d \geq 0}} \prod_{i=d'}^d \binom{-s_i}{l_i} (-1)^{s_i} (\Phi_{0z})_{p,-k} \left(\begin{matrix} z_{i_{d'+1}} \cdots z_{i_{d+1}} \\ s_{d'+1} + l_{d'+1}, \dots, s_d + l_d \end{matrix} \right) \\ \times (\Phi_{0z})_{p,-k} \left(\begin{matrix} z_{i_{d'}} \cdots z_{i_1} \\ s_{d'}, \dots, s_1 \end{matrix} \right)$$

Definition 1.3. We call the right hand side of (2) the image by Frobenius F_* of prime Taylor coefficients at O .

The reason to consider it intrinsically is that, as we are going to see, it is the emanation of a new object enabling to lift multiple harmonic sums to the motivic setting. We are going to construct a "prime multiple harmonic sum motive", which will admit, respectively, $(\text{Li } \mathcal{T})_{O, \text{prime}}$ and the right hand side of (2) as, respectively, a "Taylor period" and a " p -adic period". They happen to be related by the theorem, but we will study their properties independently. In fine, although the equalities of the theorem involve infinite sums, we will conclude that they are of geometric nature. The motive will lie in the (weight-adic) completion of the graded Hopf algebra of motivic hyperlogarithms, in which certain infinite sums are permitted. Moreover, the morphism F_* , restricted to Taylor coefficients of hyperlogarithms at limits of words on e_Z , should be seen as the realization of an element of the motivic Galois group, thus which preserves algebraic relations.

The known families of algebraic relations between hyperlogarithms, and conjecturally all of them, are homogeneous for the weight. In the language of the motivic Galois group, this is expressed as saying that the Galois group of the category of mixed Tate motives $\text{MTM}(\mathcal{O}_{\mathbb{Q}(z_1, \dots, z_r)})$ contains as a subgroup \mathbb{G}_m , which acts on motivic hyperlogarithms by $\lambda \mapsto \lambda^{\text{weight}}$.

Thus, in order to ensure that the algebraic relations between our infinite sums of p -adic multiple zeta values reflect infinite sums of algebraic relations, but also to eliminate the issues of convergence of the series in the complex case, and, finally, for generality, we will replace $\frac{1}{1-e_0} = \sum_{l \geq 0} e_0^l$ by $\frac{1}{1-\Lambda e_0}$ where Λ is a formal variable.

We continue to anticipate on the conclusion of the article and take the following notation and terminology, with w and \tilde{w} as in the statement of the theorem :

Definition 1.4. Let the (complex, p -adic and motivic) weight-adic formal series analogues of prime Taylor coefficients at O be :

$$\begin{aligned} (\text{Li } \mathcal{T})_{(O, \text{prime})}^{\mathbb{C}[[\Lambda]]}[\tilde{w}] &= (\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[\frac{1}{1 - \Lambda e_0} w \right] \\ (\text{Li } \mathcal{T})_{(O, \text{prime})}^{\mathbb{C}_p[[\Lambda]], -k}[\tilde{w}] &= (\Phi_{0z}^{-1} e_z \Phi_{0z})_{p, -k} \left[\frac{1}{1 - \Lambda e_0} w \right] \\ (\text{Li } \mathcal{T})_{(O, \text{prime})}^{\hat{\mathcal{M}}}[\tilde{w}] &= (\Phi_{0z}^{-1} e_z \Phi_{0z})_m \left[\frac{1}{1 - \Lambda e_0} w \right] \end{aligned}$$

where the hat refers to the weight-adic completion of the graded algebra of motivic hyperlogarithms.

These are the second main object of study of this paper. We will study those three variants jointly, using only the double shuffle, associator, Kashiwara-Vergne relations for complex, p -adic and motivic multiple zeta values. We will obtain our results by taking formal infinite sums of relations between multiple zeta values. This contrasts with the approach of taking Taylor coefficients evoked in §1.4.

1.6. Outline. The §2 is devoted to recall the main definitions precisely. In §2.1 we recall general facts on the pro-unipotent fundamental group. In §2.2, we recall basic facts on the spaces $M_{0,n}$; a presentation of their de Rham pro-unipotent fundamental group at its canonical base point, and the formula for connection ∇_{KZ} , with an emphasis on the case of $\mathbb{P}^1 - Z$; how multiple zeta values, hyperlogarithms, and their motivic version are defined in this setting. In §2.3, we recall the formula for the motivic Galois coaction on motivic hyperlogarithms, and the three standard families of algebraic relations.

The §3 is devoted to establish the concrete computational setting of this paper, which we will use directly for the proofs. In §3.1 we make a choice of a base-point of reference. In §3.2, we recall precisely the two relations between multiple harmonic sums and Taylor coefficients of hyperlogarithms ; we define particular operations on Taylor coefficients, with a particular focus on the degrees equal to powers of prime numbers. In §3.3, we establish the setting to deal with the infinite sums of p -adic multiple zeta values appearing in the theorem 1.2 ; namely, we define maps, consisting in modifying either the path of integration, either the sequences of differential forms to integrate, along which we will

push forward the algebraic relations between iterated integrals.

The §4 is devoted to the double shuffle relations. The statements appear in §4.1, where the main result is :

Theorem 1 (see §4.1)

- i) There are relations of double shuffle type for $(\text{Li } \mathcal{T})_O$ and $(\text{Li } \mathcal{T})_{O,\text{prime}}$.
- ii) There are relations of double shuffle type for the images by Frobenius of $(\text{Li } \mathcal{T})_{O,\text{prime}}$ and their analogues.

The part of the equations of i) concerning $(\text{Li } \mathcal{T})_{O,\text{prime}}$ is similar to the equations of ii).

The part concerning Taylor coefficients is proven in §4.2. The part concerning images by Frobenius and their analogues is proven in §4.3. The Taylor coefficient part is straightforward once the setting is established, whereas the infinite sum part is more involved.

The question of determining the analogues of associator and Kashiwara-Vergne equations requires to take a certain point of view ; associator and Kashiwara-Vergne equations do not lead directly to analogues for the prime multiple harmonic sums, as the double shuffle equations, and it is less clear a priori what the good analogues are.

In §5.1, we make a heuristic discussion explaining the point of view that we take on the analogues of associator and Kashiwara-Vergne equations. The rest of the §5 is devoted to associator equations.

We start by stating pre-associator equations and some "relative variants" in §5.2. In §5.3 and §5.4, we introduce versions of, respectively, Taylor coefficients of hyperlogarithms at the origin and of their images by Frobenius and analogues, which we denote by $(\text{Li } \mathcal{T})_{\text{Orb } \mathcal{O}}$, resp. $(\text{Li } \mathcal{T})_{\text{Orb } \mathcal{O},\text{prime}}^{\mathcal{M}}$ - they involve Taylor coefficients or their images by Frobenius at several base-points at the same time, even in the case of $\mathbb{P}^1 - \{0, 1, \infty\}$ - and explain how they satisfy variants of pre-associator relations. The theorem that follows is finally stated in §5.5.

Theorem 2 (see §5.5)

There are variants of pre-associator equations for $(\text{Li } \mathcal{T})_{\text{Orb } \mathcal{O}}$, and some $(\text{Li } \mathcal{T})_{\text{Orb } \mathcal{O},\text{prime}}^{\mathcal{M}}$.

The §6 is devoted to Kashiwara-Vergne equations, again according to the discussion of §3.4 - note that the result of theorem 3 provides an element of justification for this point of view. The §6.1 is devoted to statements :

Theorem 3 (see §6.1)

- i) There are analogues (and other that are supposed to be analogues) of Kashiwara-Vergne equations for $(\text{Li } \mathcal{T})_O$ and for $(\text{Li } \mathcal{T})_{O,\text{prime}}$.
- ii) There are relations of Kashiwara-Vergne type for the images by Frobenius of $(\text{Li } \mathcal{T})_{O,\text{prime}}$ and their analogues.

As for the part expressible purely in dimension 1, the equations i) and ii) are identical.

The part ii) is not as explicit as the other statement. It is a subject in itself concerning Kashiwara-Vergne equations themselves, and not primarily related to multiple harmonic sums, to make it explicit ; we leave it to a separate paper. The proofs for Taylor coefficients appears in §6.2, and for infinite sums in §6.3.

The §7 is dedicated to the motivic aspects, the definition of the prime multiple harmonic sum motive of §1.5 being justified the analogies observed in theorems 1 and 3, between their respective parts i) and ii).

Theorem 4 (see §7.1 for a precise statement)

Goncharov's motivic Galois coaction can be turned into a motivic coaction on the prime multiple harmonic sum motive, via its "monodromic" variant $(\Phi_{0z}^{-1})^{\mathcal{M}} e^{2i\pi e_z} \Phi_{0z}^{\mathcal{M}} \left[\frac{1}{1-\Lambda e_0} \tilde{w} \right]$.

We also define period maps and state the corresponding period conjectures. Here, the context is not \mathbb{Q} -algebras of periods and motives as usual, but weight-adic complete topological \mathbb{Q} -algebras, generated, respectively, by the prime multiple harmonic sum motive, and its formal complex and p -adic periods.

We delay to the part II the more delicate definition of the Taylor period map and its period conjecture, again in the context of complete topological algebras.

In §8 we derive the following remarks. In §8.1, motivated by computations of §7, we discuss the lifts of our results to the setting where the motivic version of $2i\pi$ is non zero, and in particular, the properties of the variant $(\Phi_{0z}^{-1})^{\mathcal{M}} e^{2i\pi e_z} \Phi_{0z}^{\mathcal{M}} \left[\frac{1}{1-\Lambda e_0} \tilde{w} \right]$. In §8.2, we explain the existence of a "shifted" prime multiple harmonic sum motive, which admits as periods the prime multiple harmonic sums with shifted upper and lower bounds. Finally, in §8.3, we discuss the variants of theorems 2 and 3 obtained by taking into account more general automorphisms of $M_{0,n}$.

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2. REVIEW OF THE PRO-UNIPOTENT FUNDAMENTAL GROUP AND ALGEBRAIC RELATIONS BETWEEN ITS PERIODS

2.1. Generalities on pro-unipotent fundamental groupoids.

2.1.1. *Generalities on pro-unipotent algebraic groups.* Let K be a field of characteristic 0. Details on the two following statements can be found in [U1], §4.2.

Proposition 2.1. The functor associating to an algebraic group over K its Lie algebra induces an equivalence of categories between the category of pro-unipotent algebraic groups and pro-nilpotent Lie algebras over K .

Proposition 2.2. Let G be a pro-unipotent algebraic group over K . There is a natural structure of Hopf algebra on the completion $\hat{\mathcal{U}}(\mathrm{Lie}(G))$ of the enveloping algebra of $\mathrm{Lie}(G)$, which is canonically isomorphic to the dual of $\mathcal{O}(G)$. This gives a canonical identification

$$G(K) = \{f \in \hat{\mathcal{U}}(\mathrm{Lie}(G)) \mid \epsilon(f) = 1, \Delta(f) = f \otimes f\}$$

where ϵ and Δ are, respectively, the counity and the coproduct of $\hat{\mathcal{U}}(\mathrm{Lie}(G))$.

Proposition 2.3. [FR] The functor of pro-unipotent completion from groups to pro-unipotent algebraic groups over \mathbb{Q} preserves split short exact sequences.

2.1.2. *Generalities on the pro-unipotent fundamental groupoid.* We recall basic facts on the motivic pro-unipotent fundamental group and its Betti and De Rham realizations. For definitions on the crystalline version, see [D], §11, §13, and [?]. We will not use the l -adic version, whose definition can also be found in [D].

2.1.2..a. de Rham realization

Let \overline{X} be a proper and smooth algebraic variety over a field K of characteristic 0, D a normal crossings divisor, and $X = \overline{X} - D$.

Definition 2.4. (Deligne, [D]) The de Rham pro-unipotent fundamental groupoid $\pi_1^{\mathrm{dR}}(X)$ is the fundamental groupoid associated to the tannakian category C^{dR} over K of vector bundles on X equipped with an integrable connection having logarithmic singularities at D , and which are unipotent.

It is a groupoid over X . Each $\pi_1^{\mathrm{dR}}(X, x, y)$ is a pro-affine scheme (by definition, the scheme of isomorphisms between the tensor functors "fiber at x " and "fiber at y "). It is a bi-torsor under the couple $(\pi_1^{\mathrm{dR}}(X, x, x), \pi_1^{\mathrm{dR}}(X, y, y))$ of pro-unipotent group schemes.

The base points are not only the rational points of X , but also the non-zero rational points of the tangent spaces at point of D to \overline{X} ([D], §15). These points are called tangential base-points. A tangential base-point at an extremity of a path is the necessary datum to define an iterated integral on this path by regularization, when the extremity of the path contains a singularity of the differential forms under the integral.

Theorem 2.5. ([D], §12) Assume that D is a sum of smooth divisors and $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$.

i) The functor $C^{dR} \rightarrow \text{Vec}_K$ which maps an object (E, ∇) to the vector space of global sections of \bar{E} on \mathbb{P}^1 is exact and is a tensor functor (§12.2).

It defines a canonical base point can of the de Rham fundamental group (§12.4, (12.4.1)).

ii) There exists a canonical isomorphism between can and each functor "fiber at x ". This defines by composition a canonical point in each $\pi_1^{un, dR}(X; y, x)$; those canonical points are compatible to the groupoid structure of $\pi_1^{un, dR}$.

iii) The canonical base point induces an equivalence between C^{dR} and the category of nilpotent representation of a Lie algebra given by an explicit presentation (§12.8)

The assumption $H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ will be satisfied in our cases. This gives the possibility to make the computations by identifying each $\pi_1^{un, dR}(X; y, x)$ to the group scheme $\pi_1^{un, dR}(X; can)$, which is explicit and independent of the base-points.

The de Rham fundamental groupoid is the universal object of its category. It is itself equipped with a canonical unipotent connection.

2.1.2..b. Betti realization

Definition 2.6. Let H be a group. The Malcev completion H^{un} of H is the universal pro-unipotent affine group scheme U over \mathbb{Q} equipped with a morphism of groups $H \rightarrow U(\mathbb{Q})$.

We now assume that K is a subfield of \mathbb{C} and take the previous hypothesis on X .

Definition 2.7. ([D], §13) The Betti pro-unipotent fundamental groupoid of $X(\mathbb{C})$ is the Malcev completion of the topological fundamental groupoid $\pi_1(X)$, in the following sense :

i) in the case of tangential base-points, the topological fundamental group is defined as follows, with $x, y \in D(\mathbb{C})$,

$$\pi_1^{top}(X(\mathbb{C}), \vec{v}_x, \vec{w}_y) = \{\gamma : [0, 1] \rightarrow \bar{X}(\mathbb{C}) \mid \gamma([0, 1]) \subset X, \gamma'(0) = \vec{v}, \gamma'(1) = \vec{w}\} / \sim$$

where \sim is the natural associated homotopy equivalence (in particular, it preserves the values of $\gamma'(0)$ and $\gamma'(1)$).

ii) the Malcev completion is applied to the groups $\pi_1^{top}(X(\mathbb{C}); x)$, and then to the sets $\pi_1^{top}(X(\mathbb{C}); y, x)$ using that they are $\pi_1^{top}(X(\mathbb{C}))(X; x)$ -torsors

Theorem 2.8. ([D], §10) $\pi_1^{un, B}(X/\mathbb{C})$ is also the fundamental groupoid associated to the Tannakian category over \mathbb{Q} of local systems over $X(\mathbb{C})$ which are unipotent (i.e. iterated extensions of the trivial object).

This category is equivalent to the category of functors $C \rightarrow \text{Vect}_{\mathbb{Q}}$ where C is the category having as objects the base-points of X , as sets of morphisms from x to y , for base-points x, y , the set $\pi_1(X(\mathbb{C}), y, x)$.

2.1.2..c. Betti-de Rham comparison isomorphism

The comparison between the two versions is a reformulation of Chen's theorem on iterated path integrals [Ch]. It is a variant of the Riemann-Hilbert correspondence.

Theorem 2.9. With the assumptions of the definitions, we have isomorphisms of groupoids :

$$\text{comp} : \pi_1^{\text{un}, \text{B}}(X)(\mathbb{C}) \xrightarrow{\sim} \pi_1^{\text{un}, \text{dR}}(X)(\mathbb{C})$$

2.1.2..d. Motivic version

A notion of motivic pro-unipotent fundamental group is defined in [D], §13, directly in terms of the realizations, their respective structures and their isomorphisms of comparison. It applies to the following situation : S an open of the spectrum of the ring of integers of a number field k , which is étale over $\text{Spec}(\mathbb{Z})$; \overline{X}_S a proper and smooth variety on S ; D_S a normal crossings divisor of \overline{X}_S and $X_S = \overline{X}_S - D_S$; with the assumption that the general fiber X of X_S over k is geometrically connected, and that $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$.

The motivic pro-unipotent fundamental group in this setting is a pro-unipotent scheme in the Tannakian category of systems of realizations over S introduced in [D] §7.

A second notion is established in [G], via a relation between the pro-unipotent fundamental group of X and the relative cohomology of powers of X .

The theorems 4.1, 4.3 and 4.4 of [G] express, respectively, the Betti, l -adic and de Rham pro-unipotent groupoids of certain varieties (after unpublished work of Beilinson, for the Betti case) as equal to the cohomology of certain complexes, the equality being compatible to the natural structures on both sides. The Betti case applies to a connected topological manifold ; the l -adic case applies to a regular variety X over a field F , with the étale l -adic cohomology with coefficients in \mathbb{Q}_l equipped with its $\text{Gal}(\overline{F}/F)$ -action. The de Rham case applies to a regular variety over a field F of characteristic zero. Finally, in the de Rham case, the equality is compatible with the mixed Hodge structures on both sides.

A complex in the category of algebraic varieties, similar to the ones appearing in the theorems 4.1, 4.3 and 4.4 of [G], defines an object in a triangulated category of motives \mathcal{DM}_F ([G] §4.3) ; this object is called the motivic torsor of paths ; its Betti, de Rham, Hodge and l -adic realizations coincide with the torsors of paths of the realizations of the pro-unipotent fundamental group of [D].

Using the previous facts, a third construction is given in [DG] for a unirational variety on a number field. Consider the category $\text{MTM}(k)$ of mixed Tate over a number field k , and the category $\text{MTM}(O_S)$ of mixed Tate motives on a ring of S -integers where S a finite set of places of k .

When X is equal to $\mathbb{P}^1 - Z$ over a number field k , the motivic fundamental group of X ([DG], §4) is a groupoid on X in schemes in the category $MT(k)$ - in the sense of schemes on an Tannakian category as in [DG], §2. It lifts the systems of realization constructed in [D] using the results of [G].

There is a canonical fiber functor ω on $\text{MTM}(O_S)$; let $G^\omega = \text{Aut}^\otimes \omega$ be the associated Tannakian group. There also exist Betti, de Rham and Hodge realization functors ; the scalar extension to k of ω is canonically isomorphic with the de Rham fiber functor on $\text{MTM}(k)$ ([DG], proposition 2.10).

By the relation between the pro-unipotent fundamental group and cohomology, investigated in [DG], §3, and by the hypothesis on X , the group G^ω acts on the de Rham fundamental groupoid of X .

For our purposes, it is enough to consider that the motivic fundamental group is the data of the de Rham pro-unipotent fundamental group equipped with the Galois action of the Tannakian group G^ω .

We will use this last point of view, and the formula for the action of G^ω on hyperlogarithms computed in [G], §6, in the Hodge realization, which is possible because the Hodge realization is fully faithful, by [DG] proposition 2.14.

2.2. Description in the case of $M_{0,n}$.

2.2.1. The spaces $M_{0,n}$ and $\overline{M}_{0,n}$.

2.2.1.a. The spaces $M_{0,n}$

Let again

$$M_{0,n+3} = \{(x_1, x_2, x_3, x_4) \in (\mathbb{P}^1)^4 \mid x_1, x_2, x_3, x_4 \text{ distincts}\} / \text{PGL}_2$$

The unique homography sending $(x_{n+1}, x_{n+2}, x_{n+3}) \mapsto (0, 1, \infty)$ yields an identification

$$M_{0,n+3} \simeq \{(y_1, y_2, \dots, y_n) \in (\mathbb{P}^1 - \{0, 1, \infty\})^n \mid \text{for all } i, j, y_i \neq y_j\}$$

where the y_i 's are called simplicial coordinates and are given by cross-ratios

$$y_i = \frac{x_i - x_{n+1}}{x_i - x_{n+3}} \frac{x_{n+2} - x_{n+3}}{x_{n+2} - x_{n+1}}$$

In particular, we have a natural identification $M_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$. Let also the cubic coordinates be defined by, for all $i \in \{1, \dots, n\}$,

$$y_i = c_i \dots c_n$$

Let $\overline{M}_{0,n+3}$ be the Deligne-Mumford compactification of $M_{0,n+3}$. It is, by definition, a smooth and projective variety containing $M_{0,n+3}$ such that $\partial \overline{M}_{0,n+3} = \overline{M}_{0,n+3} - M_{0,n+3}$ is a normal crossings divisor. The divisor at infinity has a natural stratification whose combinatorial description is explained for example in [GM].

We have $\overline{M}_{0,4} = \mathbb{P}^1$ and, as soon as $n \geq 2$, there are several ways to define $\overline{M}_{0,n+3}$. In the case of $M_{0,5}$, a simple one is the following : view $M_{0,5}$ in simplicial coordinates as a subvariety of $(\mathbb{P}^1)^2$; then, $\overline{M}_{0,5}$ is obtained by blowing up $(\mathbb{P}^1)^2$ at the three points where $(\mathbb{P}^1)^2 - M_{0,5}$ is not normal crossings, namely, $(0, 0)$, $(1, 1)$ and (∞, ∞) .

We will use below that :

Proposition 2.10. [FR] The $M_{0,n+3}$'s are $K(\pi, 1)$ spaces.

An important result is :

Theorem 2.11. (Bruno-Mella) ([BM]) Assume that $n \geq 2$. The group of automorphism of $\overline{M}_{0,n+3}$ is the symmetric group S_{n+3} , of the permutations of the $n + 3$ canonical coordinates.

2.2.1.b. The forgetful maps $M_{0,n+m} \rightarrow M_{0,n}$

Let $r \in \mathbb{N}^*$. The fibers of $M_{0,r+4} \rightarrow M_{0,r+3}$ are curves $X_Z = \mathbb{P}^1 - Z$ with Z a finite subset of \mathbb{P}^1 , of cardinal $r+3$. The fibers of $M_{0,r+5} \rightarrow M_{0,r+3}$ are the $X_Z^2 - \Delta_Z$, where Δ_Z is the diagonal of X_Z^2 .

We will denote the elements of such a Z by $Z = \{0, 1, \infty, z_1, \dots, z_r\}$ (assuming without loss of generality that Z contains $\{0, 1, \infty\}$) with, moreover, $z_0 = 0$ and $z_{r+1} = 1$.

Proposition 2.12. (Bruno-Mella) [BM] The forgetful maps are the only morphisms $\overline{M}_{0,m+3} \rightarrow \overline{M}_{0,n+3}$, with $m > n$.

The fibrations $M_{0,m+3} \rightarrow M_{0,n+3}$ induce long exact sequences in homotopy ; using that $M_{0,n+3}$'s are $K(\pi, 1)$ spaces, and the exactness property of the pro-unipotent completion functor, both evoked above, we obtain :

Lemma 2.13. [FR] Given a forgetful map $M_{0,m+3} \rightarrow M_{0,n+3}$, and its fiber F_{m-n} , we have a short split exact sequence :

$$1 \rightarrow \pi_1^{un,B}(F_{m-n})(\mathbb{C}) \rightarrow \pi_1^{un,B}(M_{0,m+3}(\mathbb{C})) \rightarrow \pi_1^{un,B}(M_{0,n+3}(\mathbb{C})) \rightarrow 1$$

2.2.1.c. Particular tangential base-points on $\mathbb{P}^1 - Z$

We will consider the base-point $\vec{1}_0$ on $\mathbb{P}^1 - Z$ and, more generally, vectors ± 1 at the points of Z .

In the cyclotomic case of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$, let also the set of tangent vectors :

$$\mathfrak{T}_N = \{\pm \vec{\eta}_\xi \mid \eta^N = 1, \xi \in \{0, \mu_N, \infty\}\}$$

In the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, this is a set of six tangential base-points :

$$\mathfrak{T}_1 = \{\vec{1}_0, -\vec{1}_0, \vec{1}_1, -\vec{1}_1, \vec{1}_\infty, -\vec{1}_\infty\}$$

The subset of elements of $\text{Aut}(\mathbb{P}^1)$ which preserve $\{0, 1, \infty\}$ identifies to S_3 and acts simply transitively on \mathfrak{T} . Its induced action on $\pi_1^{\text{dR}}(X; \mathfrak{T}, \mathfrak{T})$ is given, for $s \in S_3$ and $f(e_a, e_b) \in \pi_1^{\text{dR}}(X; b, a)(R)$, by $s_*(f) = f(e_{s(a)}, e_{s(b)}) \in \pi_1^{\text{dR}}(X; s(b), s(a))(R)$. We denote such an element s by $\sigma_{s(0), s(1)}$.

2.2.2. Presentation of the de Rham realization and its connection. The hypothesis $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ is satisfied in the following cases, thus the description of the de Rham pro-unipotent fundamental group provided by [D] §12 applies.

We state a presentation for $\pi_1^{un,dR}$ at the canonical base point can in several examples, and a formula for the canonical connection ∇_{KZ} , according to the point of view of [D], §7.30 - §7.32, and under the identification of the fibers of $\pi_1^{un,dR}$ with the fiber at the canonical base point.

2.2.2.a. The case of $M_{0,n}$

Lemma 2.14. $\text{Lie}(\pi_1^{\text{dR}}(M_{0,r+3}, can))$ is the pro-nilpotent Lie algebra with generators e_{ij} , $1 \leq i \neq j \leq r+3$, and relations $e_{ij} = e_{ji}$, for all i , $\sum_{j=1}^n e_{ij} = 0$, and for all i, j, k, l such that $\sharp\{i, j, k, l\} = 4$, $[e_{ij}, e_{kl}] = 0$.

These relations imply the relation $[e_{ij} + e_{jk}, e_{ik}] = 0$ for i, j, k pairwise distincts.

Lemma 2.15. The universal connection on fundamental torsors of paths is given in canonical coordinates by :

$$\nabla_{\text{KZ}} : f \mapsto f^{-1}(df - \sum_{1 \leq i < j \leq n} e_{ij} d \log(x_i - x_j) f)$$

Let $\begin{cases} C_{0,u} = \sum_{u \leq i < j \leq n} e_{ij} \text{ for all } u \\ C_{1,v,v'} = -e_{v-1,v'} \text{ for } 2 \leq i \leq j \\ C_{1,1,v'} = -e_{v',n-1} \text{ for } 1 \leq i \leq j \end{cases}$ Then, in cubic coordinates we have :

$$\nabla_{\text{KZ}} : f \mapsto f^{-1}(df - \left(\sum_{u=1}^r \frac{dc_u}{c_u} C_{0,u} + \sum_{1 \leq v \leq v' \leq n} \frac{d(c_v \dots c_{v'})}{c_v \dots c_{v'} - 1} C_{1,v,v'} \right) f)$$

In other terms, the connection coincides (up to the factor f^{-1}) with the known Knizhnik-Zamolodchikov differential system on a trivial bundle over $\text{Lie}(\pi_1^{\text{dR}}(M_{0,r+3}, \text{can}))$, appearing for example in [Dr], §2 ; this justifies the terminology.

2.2.2.b. The case of $X_Z = \mathbb{P}^1 - Z$

We take again the same notation for the elements of Z . What follows is, for example, a consequence of [D] §12. Note that the pro-unipotent fundamental group is defined over \mathbb{Q} although X_Z is defined on a number field.

Lemma 2.16. The Lie algebra $\text{Lie}(\pi_1^{\text{dR}}(X_Z; \text{can}))$ is the pro-nilpotent free Lie algebra generated by $H^{1,dR}(X_Z)^\vee$, i.e. generated by the formal residues $e_{z_0}, \dots, e_{z_{r+1}}, e_\infty$ subject to the relation $e_0 + \dots + e_{z_{r+1}} + e_\infty = 0$.

Definition 2.17. Let $\mathcal{H}_{\mathfrak{m}}(e_Z)$ be the \mathbb{Q} -vector space $\mathbb{Q}\langle (e_{z_i})_{i=0,\dots,r+1} \rangle = \mathbb{Q}\langle e_Z \rangle$, freely generated by words over e_Z , including the empty word. It is graded by the length of words called the "weight" of words. It is a Hopf algebra, called the shuffle Hopf algebra, endowed with :

i) The shuffle product \mathfrak{m} defined by, for all words $u_1 \dots u_m, u_{m+1} \dots u_{m+m'}$ over e_Z :

$$(u_1 \dots u_m) \mathfrak{m} (u_{m+1} \dots u_{m+m'}) = \sum_{\substack{\sigma \text{ permutation of } \{1, \dots, m+m'\} \\ \sigma(1) < \dots < \sigma(m) \\ \sigma(m+1) < \dots < \sigma(m+m')}} u_{\sigma^{-1}(1)} \dots u_{\sigma^{-1}(m+m')}$$

ii) The deconcatenation coproduct $\Delta_{\text{dec}} : u_1 \dots u_m \mapsto \sum_{k=0}^r u_1 \dots u_k \otimes u_{k+1} \dots u_m$

iii) the counit ϵ equal to the augmentation morphism

iv) the antipode $S : u_m \dots u_1 \mapsto (-1)^m u_1 \dots u_m$.

By the generalities of §2.1.1 on pro-unipotent fundamental groups :

Corollary 2.18. The Hopf algebra $\mathcal{O}(\pi_1^{\text{dR}}(X_Z; \text{can}))$ is the shuffle Hopf algebra $\mathcal{H}_{\mathfrak{m}}(e_Z)$ over \mathbb{Q} associated with the alphabet $e_Z = \{e_{z_0}, \dots, e_{z_{r+1}}\}$

This implies immediately :

Corollary 2.19. The completed graded dual $\widehat{\mathcal{H}_{\mathfrak{m}}(Z)}^\vee$ of $\mathcal{H}_{\mathfrak{m}}(Z)$ identifies with the non-commutative algebra of series over $(e_{z_i})_{i=0,\dots,r}$; the coproduct, dual to \mathfrak{m} , is called the shuffle coproduct and denoted by $\Delta_{\mathfrak{m}}$. It is the unique linear multiplicative map, continuous for the $\ker(\epsilon)$ -adic topology, satisfying $\Delta_{\mathfrak{m}}(e_i) = e_i \otimes 1 + 1 \otimes e_i, i = 0, 1$.

Again by the generalities on pro-unipotent fundamental group of §2.1.1 :

Fact 2.20. We have :

i) The points of the group $\pi_1^{\text{dR}}(X; \eta)$ are the grouplike series :

$$\pi_1^{\text{dR}}(X_Z; \eta)(R) = \{f \in R\langle\langle e_Z \rangle\rangle \text{ s.t. } \Delta_{\mathfrak{m}}(f) = f \otimes f, \epsilon(f) = 1\}$$

ii) We have

$$\text{Lie}(\mathcal{H}_{\mathfrak{m}}^{\vee}(Z)) \otimes_{K(Z)} R = \{f \in R\langle\langle e_Z \rangle\rangle \text{ s.t. } \Delta_{\mathfrak{m}}(f) = f \otimes 1 + 1 \otimes f\}$$

Now we describe the connection :

Lemma 2.21. On the torsor $(\pi_1^{\text{dR}}(X_Z; z, \vec{1}_0)_z)$, of paths starting at the tangential base-point $\vec{1}_0$ (i.e. $\vec{1}$ at 0), trivialized at $\vec{1}_0$, the connection is

$$\nabla_{KZ} : f \mapsto f^{-1} \left(df - \left(\sum_{i=0}^{r+1} e_{z_i} \frac{dz}{z - z_i} \right) f \right)$$

We will also recall other facts on the shuffle Hopf algebra in the paragraph on algebraic relations (§2.3).

2.2.2.c. The case of $X_Z^2 - \Delta_Z$

The pro-unipotent fundamental group can be computed by the exact sequence of pro-unipotent fundamental groups of lemma 2.13. It gives a presentation of $\pi_1^{\text{un}, \text{dR}}(X_Z^2 - \Delta_Z, \text{can})$ involving formal variables A_{1i} , A'_{1i} and A'' . The connection ∇_{KZ} associated with $\pi_1^{\text{un}, \text{dR}}((\mathbb{P}^1 - Z)^2 - \Delta)$ can be written in cubic coordinates as

$$f \mapsto f^{-1} \left(df - \left(\sum_{i=0}^{r+1} \frac{dc_1}{c_1 - z_i} A_{1i} + \sum_{i=0}^{r+1} \frac{d(c_1 c_2 - c_1)}{c_1 c_2 - z_i} A'_{1i} + \frac{d(c_1 c_2 - c_1)}{c_1 c_2 - c_1} A'' \right) f \right)$$

Note that we have $\frac{d(c_1 c_2 - c_1)}{c_1 c_2 - c_1} = \frac{dc_2}{c_2 - 1} + \frac{dc_1}{c_1}$.

2.2.3. Multiple zeta values and hyperlogarithms in the context of the fundamental group.

2.2.3.a. As Betti-de Rham periods

We can now restate the definition of multiple zeta values and hyperlogarithms in this context, following [DG], and implicitly [Dr], §2. Let the straight path from 0 to 1, $\gamma : t \in [0, 1] \mapsto [0, 1]$, as an element of $\pi_1^{\text{top}}(M_{0,4}(\mathbb{C}), \vec{1}_0, -\vec{1}_1)$; following [DG], §5.16, it induces an element $\text{dch} \in \pi_1^{\text{un}, \text{B}}(M_{0,4}(\mathbb{C}), \vec{1}_0, -\vec{1}_1)(\mathbb{R})$, the "droit chemin".

Definition 2.22. ([DG], §5.16) Let the image of dch by the Betti-de Rham comparison of Theorem 2.9 :

$$(5) \quad \Phi := \text{comp}(\text{dch}) \in \pi_1^{\text{un}, \text{dR}}(M_{0,4}; \vec{1}_0, -\vec{1}_1)(\mathbb{R})$$

In other words Φ is the integration on ∇_{KZ} along dch . This is equivalent to the formula for multiple zeta values as iterated integrals given in §1.2 : using the notations of §1.3 for non commutative formal power series, we have, for indices such that $s_d \geq 2$:

$$(6) \quad \zeta(s_d, \dots, s_1) = (-1)^d \Phi[e_0^{s_d-1} e_1 \dots e_0^{s_1-1} e_1]$$

Similarly, each Φ_{0z} is the image by the Betti-de Rham comparison of an element defined by a topological path, usually the straight path from 0 to z .

We will mostly use words of differential forms giving iterated integrals that are regular at 0. They are of the form $w = e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}$, with $i_1, \dots, i_d \in Z - \{0, \infty\}$.

2.2.3.b. The p -adic version

The p -adic analogues are defined through the Frobenius action on the p -adic points of the de Rham fundamental group. The definitions that we use are the ones of Deligne-Goncharov [DG] §5.28, generalized to all the powers of Frobenius and curves $\mathbb{P}^1 - Z$ such that $\mathcal{O}_{\mathbb{Q}_p(z_1, \dots, z_r)}$ is absolutely non ramified (see our papers [J1], [J2]). A different definition is due to Furusho [F1], [F2].

2.2.3.c. The motivic versions

The motivic hyperlogarithms have been defined in [G], §5, using the motivic torsor of path constructed in §4.3 of the same paper : definition 5.5 of [G]. They are actually constructed as framed Hodge-Tate structures. We recall again that the Hodge realization functor of categories of mixed Tate motives is fully faithful.

What we will use is only the formula for the motivic co-action, that we recall below.

2.3. Formula for the motivic coproduct and three families of algebraic relations. Here are now the basis of the algebraic theory of multiple zeta values and hyperlogarithms in this context.

2.3.1. The motivic coproduct. We write the formula for the action of the mixed Tate motivic Galois group G_ω on the de Rham pro-unipotent fundamental group, as the coaction of the Hopf algebra $\mathcal{O}(G_\omega)$ on the algebra of functions $\mathcal{O}(\pi_1^{un, dR}(X_Z, \vec{1}_0, -\vec{1}_1))$ - identified canonically to $\mathcal{O}(\pi_1^{un, dR}(X_Z, can)) \simeq \mathcal{H}_m(e_Z)$.

The Tannakian group G_ω is an affine (pro)-algebraic group equal to a semi-direct product, where U_ω is a pro-unipotent affine group scheme.

$$G_\omega = \mathbb{G}_m \ltimes U_\omega$$

- The coaction of \mathbb{G}_m is

$$\lambda \mapsto \text{multiplication by } \lambda^{\text{weight}}$$

The presence of this action means that the algebraic relations between hyperlogarithms are and the general conjecture that all relations among hyperlogarithms remain true in the motivic setting implies then that all algebraic relations among hyperlogarithms are homogeneous for the weight.

- The formula for the action of the pro-unipotent part U_ω is provided by what follows. Following [G], let $I(a_{n+1}, a_n, \dots, a_1; a_0)$ be an iterated integral $\int_{a_0}^{a_{n+1}} \omega_{a_n} \dots \omega_{a_1}$. Let $\tilde{I}(a_{n+1}, a_n, \dots, a_1; a_0)$ be the Hodge-Tate structure lifting it ([G], definition 5.5).

Theorem 2.23. (Goncharov, [G], theorem 6.4, [Br3]) i) The coproduct on \tilde{I} is given by :

$$(7) \quad \Delta \tilde{I}(a_{n+1}; a_n, \dots, a_1; a_0) \\ = \sum_{0=i_0 < i_1 < \dots < i_k < i_{k+1}=n} \tilde{I}(a_{n+1}; a_{i_k}, \dots, a_{i_1}; a_0) \otimes \prod_{l=0}^k \tilde{I}(a_{i_{p+1}}; a_{i_{p+1}-1} \dots a_{i_p+1}; a_{i_p})$$

ii) This formula stays true for Brown's version of the same objects as in [Br3], for which $(2i\pi)^{\mathcal{M}} \neq 0$.

We choose $a_0 = \vec{I}_0$ and $a_{n+1} = -\vec{I}_1$. Note that the formula involves different base-points the same time ; this is only an appearance, because of the invariance of these objects by affine transformations of an a_i written in [G], §1 ; and this coproduct can be thought of as a map

$$\mathcal{O}(\pi_1^{un,dR}(X_Z, \vec{I}_0, -\vec{I}_1)) \rightarrow \mathcal{O}(\pi_1^{un,dR}(X_Z, \vec{I}_0, -\vec{I}_1)) \otimes \mathcal{O}(\pi_1^{un,dR}(X_{Z'}, \vec{I}_0, -\vec{I}_1))$$

for another Z' . Composing it with, on the right hand side, the map $\mathcal{O}(\pi_1^{dR}(X_{Z'}, \vec{I}_0, -\vec{I}_1)) \rightarrow \mathcal{O}(U_\omega)$ given by the action of $\mathcal{O}(U_\omega)$ on the canonical path 1 - this is the path provided by the canonical de Rham base point *can* - we obtain the coaction :

$$\mathcal{O}(\pi_1^{un,dR}(X_Z, \vec{I}_0, -\vec{I}_1)) \rightarrow \mathcal{O}(\pi_1^{un,dR}(X_Z, \vec{I}_0, -\vec{I}_1)) \otimes \mathcal{O}(U_\omega)$$

2.3.2. Double shuffle relations and associated structures.

2.3.2.a. The shuffle product

We already have defined the shuffle product \mathfrak{m} in §1.2. It is a general definition which applies to words over any alphabet.

It will be useful, for computations, to consider the alternative definition which is recursive on the weight. Let $i \in \{0, \dots, r+1\}$.

Definition 2.24. Let $\partial_{e_{z_i}}$, resp. $\tilde{\partial}_{e_{z_i}}$ be the linear application $\mathbb{Q}\langle e_Z \rangle \rightarrow \mathbb{Q}\langle e_Z \rangle$ defined on words by $\left\{ \begin{array}{l} \partial_{e_{z_i}}(e_{z_i}w) = w \\ \partial_{e_{z_i}}(e_{z_j}w) = 0 \text{ if } i \neq j \end{array} \right.$ resp. $\left\{ \begin{array}{l} \tilde{\partial}_{e_{z_i}}(we_{z_i}) = w \\ \tilde{\partial}_{e_{z_i}}(we_{z_j}) = 0 \text{ if } i \neq j \end{array} \right.$

We have for all $w \in \mathbb{Q}\langle e_Z \rangle$, $w = \sum_{i=0}^r e_{z_i} \partial_{e_{z_i}}(w) = \sum_{i=0}^r \tilde{\partial}_{e_{z_i}}(w) e_{z_i}$

Lemma 2.25. The shuffle product is the unique bilinear map $\mathbb{Q}\langle e_Z \rangle \times \mathbb{Q}\langle e_Z \rangle \rightarrow \mathbb{Q}\langle e_Z \rangle$ which makes the $\partial_{e_{z_i}}$'s, (resp. the $\tilde{\partial}_{e_{z_i}}$'s) into derivations. In other terms it is defined by induction on the weight by the formulas

$$(e_{i_1}w_1)\mathfrak{m}(e_{i_2}w_2) = e_{i_1}(w_1\mathfrak{m}e_{i_2}w_2) + e_{i_2}(e_{i_1}w_1\mathfrak{m}w_2) \\ (w_1e_{i_1})\mathfrak{m}(w_2e_{i_2}) = (w_1\mathfrak{m}w_2e_{i_2})e_{i_1} + (w_1\mathfrak{m}w_2e_{i_1})e_{i_2}$$

We will also use what follows which is a general fact on Hopf algebras :

Lemma 2.26. Let $h \in \text{Lie}(\mathcal{H}_{\mathfrak{m}}^\vee) \otimes_{\mathbb{Q}} R$:

$$(8) \quad (S^\vee)h = -h$$

If f is a function on $\mathbb{Q}\langle e_Z \rangle$ satisfying for all words w , that $f[w\mathfrak{m}e_0] = f[w]f[e_0]$, then, for all $s_d, \dots, s_1 \in \mathbb{N}^*$, T, U_1, \dots, U_d formal variables, $i_d, \dots, i_1 \in \{1, \dots, r\}$:

$$(9) \quad f\left[\frac{e_0^{s_d-1}}{(1-U_de_0)^{s_d}}e_{z_{i_d}} \cdots \frac{e_0^{s_1-1}}{(1-U_1e_0)^{s_1}}e_{z_{i_1}} \frac{1}{1-Te_0}\right] \\ = f\left[\frac{e_0^{s_d-1}}{(1-(U_d-T)e_0)^{s_d}}e_{z_{i_d}} \cdots \frac{e_0^{s_1-1}}{(1-(U_1-T)e_0)^{s_1}}e_{z_{i_1}}\right]e^{f[e_0]T}$$

2.3.2.b. The series shuffle product $*$

We recall first the definition of the series shuffle product associated with $\mathbb{P}^1 - \{0, 1, \infty\}$, and then the general one associated with a general $\mathbb{P}^1 - Z$.

Let us consider the alphabet $Y = \{(y_n)_{n \in \mathbb{N}}\}$ and the following bijection between the set of words on Y and the set of words w on $\{e_0, e_1\}$ such that $\tilde{\partial}_{e_1}(w) = 0$:

$$e_0^{s_d-1}e_1 \dots e_0^{s_1-1}e_1 \leftrightarrow y_{s_d} \dots y_{s_1} \quad (d \in \mathbb{N}^*, \quad s_d, \dots, s_1 \in \mathbb{N}^*)$$

There is a way of expressing a product $\{(n_1, \dots, n_d) \in (\mathbb{N}^*)^d | 0 < n_1 < \dots < n_d\} \times \{(n'_1, \dots, n'_{d'}) \in (\mathbb{N}^*)^{d'} | 0 < n'_1 < \dots < n'_{d'}\}$ as a disjoint union of subsets of $(\mathbb{N}^*)^{d+d'}$ defined by strict or large inequalities between the n_i 's and the n'_j 's. For example $(\mathbb{N}^*)^2 = \{(n, n') | n = n'\} \amalg \{(n, n') | n < n'\} \amalg \{(n, n') | n > n'\}$.

Definition 2.27. (see [H]) The quasi-shuffle or series shuffle graded Hopf algebra \mathcal{H}_* is the \mathbb{Q} -vector space $\mathbb{Q}\langle (y_s)_{s \in \mathbb{N}^*} \rangle$ of words over Y , including the empty word $y_0 = 1$, graded by the length of words. It is endowed with

i) the quasi-shuffle product $*$, defined recursively by, for w_1, w_2 words, and $s, s' \in \mathbb{N}^*$,

$$y_s w_1 * y_{s'} w_2 = y_s (w_1 * y_{s'} w_2) + y_{s'} (y_s w_1 * w_2) + y_{s+s'} (w_1 * w_2)$$

Each word appearing in the expression of $w_1 * w_2$ as sum of words is called a "series shuffle element" of (w_1, w_2) .

ii) the deconcatenation coproduct Δ_{dec} relative to words in the y_s 's

iii) the counit ϵ equal to the augmentation morphism

iv) and the antipode given by the two following formulae

$$z_{s_d, \dots, s_1} = \sum_{\substack{1 \leq l \leq d \\ 1 = i_1 < i_2 < \dots < i_l = d}} y_{\sum_{i=i_l}^{i_{l+1}=d} s_i} \cdots y_{\sum_{i=i_1}^{i_2} s_i}$$

Then

$$S(y_{s_d} \dots y_{s_1}) = (-1)^d z_{s_1, \dots, s_d} = \sum_{\substack{l \geq 1 \\ y_{s_d} \dots y_{s_1} = w_l \dots w_1}} (-1)^l w_l * \dots * w_1$$

Fact 2.28. The completed dual $\widehat{\mathcal{H}}_*^\vee$ of \mathcal{H}_* is the non-commutative algebra of series $\mathbb{Q}\langle\langle (y_s)_{s \in \mathbb{N}^*} \rangle\rangle$, equipped with the (continuous) coproduct Δ_* , satisfying $\Delta_*(y_n) = \sum_{k=0}^n y_k * y_{n-k}$.

Now we define the generalized series shuffle product. Let Z be a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$ containing $\{0, 1, \infty\}$. Let Y_Z be the alphabet $\{y_{s,z} | s \in \mathbb{N}^*, z \in Z - \{\text{infy}\}\}$. We

identify words on Y_Z to the words w on e_Z such that $\tilde{\partial}_{e_0}(w) = 0$ via the correspondence

$$y_{s_d, z_d} \cdots y_{s_1, z_1} \leftrightarrow e_0^{s_d-1} e_{z_{i_d}} \cdots e_0^{s_1-1} e_{z_{i_1}}$$

Definition 2.29. Let $\mathcal{H}_*(e_Z)$ be the \mathbb{Q} -vector space freely generated by words over Y_Z , including the empty word 1.

Notation 2.30. If Z, Z' are two finite subsets of $\mathbb{P}^1(\overline{\mathbb{Q}})$ containing $\{0, 1, \infty\}$, what we will denote by ZZ' is simply their product set, i.e. $ZZ' = \{zz', z \in Z, z' \in Z'\}$.

The generalized shuffle product is a map $\mathcal{H}_*(Z) \times \mathcal{H}_*(Z') \rightarrow \mathcal{H}_*(ZZ')$.

For convenience we extend slightly $\mathcal{H}_*(Z)$ and $\mathcal{H}_*(Z')$.

Definition 2.31. i) Let $\mathcal{H}_*^{ext}(e_Z)$ be the vector space freely generated by the empty sequence and the sequences

$$\begin{pmatrix} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix} = (z_{i_{d+1}}, s_d, \dots, z_{i_2}, s_1, z_{i_1})$$

with $s_i \in \mathbb{N}^*$ and $i_j \in \{1, \dots, r\}$

ii) Let the injective linear map $i : \mathcal{H}_*(e_Z) \hookrightarrow \mathcal{H}_*^{ext}(e_Z)$ defined by $y_{s_d, z_{i_d}} \cdots y_{s_1, z_{i_1}} \mapsto (1, s_d, z_{i_d}, \dots, s_1, z_{i_1})$ and let $r : \mathcal{H}_*^{ext}(e_Z) \hookrightarrow \mathcal{H}_*(e_Z)$ be defined by $(z_{i_{d+1}}, s_d, z_{i_d}, \dots, s_1, z_{i_1}) \mapsto (1, s_d, z_{i_d}/z_{i_{d+1}}, \dots, s_1, z_{i_1}/z_{i_{d+1}})$.

Definition 2.32. We define the bilinear generalized shuffle product

$$* : \mathcal{H}_*^{ext}(e_Z) \times \mathcal{H}_*^{ext}(e_{Z'}) \rightarrow \mathcal{H}_*^{ext}(ZZ')$$

as the unique bilinear map given on couples of words as follows. For two sequences : $w = (z_{i_{d+1}}, s_d, \dots, z_{i_2}, s_1, z_{i_1})$, $w' = (z_{j_{d'+1}}, t_{d'}, \dots, z_{i_2}, t_1, z_{i_1})$, the series shuffle $w * w'$ is the sum of all the series shuffle elements, where the series shuffle elements are defined as below, inductively on the length, using the case where $Z = \{0, 1, \infty\}$.

For each series shuffle element $(u_{d''}, \dots, u_1)$ of $((s_d, \dots, s_1), (t_{d'}, \dots, t_1))$, the set of shuffle elements

$(z_{i_{a_{d''}+1}} z_{j_{b_{d''}+1}}, u_{d''}, \dots, z_{i_{a_2}} z_{j_{b_2}}, u_1, z_{i_{a_1}} z_{j_{b_1}})$ be defined by induction as follows :

i) $a_1 = b_1 = 1$

ii) for $i \in \{1, \dots, d'' - 1\}$:

$$\begin{cases} \text{if } u_i = s_l, l \in \{1, \dots, d\}, \text{ then } (a_{i+1}, b_{i+1}) = (a_i + 1, b_i) \\ \text{if } u_i = t_{l'}, l' \in \{1, \dots, d'\}, \text{ then } (a_{i+1}, b_{i+1}) = (a_i, b_i + 1) \\ \text{if } u_i = s_l + t_{l'}, (l, l') \in \{1, \dots, d\} \times \{1, \dots, d'\}, \text{ then } (a_{i+1}, b_{i+1}) = (a_i + 1, b_i + 1) \end{cases}.$$

Definition 2.33. Let $\hat{\mathcal{H}}_*^\vee(e_Z)$ be the coalgebra dual to $\mathcal{H}_*(e_Z)$, equipped with the series shuffle coproduct Δ_* .

It identifies to a \mathbb{Q} -vector subspace of $\hat{\mathcal{H}}_{\text{in}}^\vee(e_Z)$ via the correspondence between words on Y_Z and words on e_Z , and the notation with brackets for coefficients of formal series applies to this case.

2.3.2.c. The double shuffle equations

The shuffle equation can be defined relatively to any pro-unipotent algebraic group. In our context we need only a particular case.

Definition 2.34. i) The shuffle equation is an equation to which are subject elements of $\hat{\mathcal{H}}_{\mathfrak{m}}^{\vee}(e_Z) \otimes_{\mathbb{Q}} R$, with R a \mathbb{Q} -algebra. It is the condition that we have $\Delta_{\mathfrak{m}}(f) = f \otimes f$, i.e. for all words w, w' , $f[w\mathfrak{m}w'] = f[w]f[w']$.
ii) The series shuffle equation is an equation to which are subjects elements of $\hat{\mathcal{H}}_{\mathfrak{m}}^{\vee}(e_Z) \otimes_{\mathbb{Q}} R$, with R a \mathbb{Q} -algebra ; it is the equation $\Delta_{\mathfrak{m}}(f) = f \otimes f$. It is equivalent to : for all words w, w' , $f[w * w'] = f[w]f[w']$.
iii) A couple $(f_{\mathfrak{m}}, f_*)$ is said to satisfy the regularized double shuffle equation if $f_{\mathfrak{m}}$ satisfies the shuffle equation, f_* satisfies the series shuffle equation, and $f_{\mathfrak{m}}[w] = f_*[w]$ for words over $Y_Z, y_{s_d, z_d} \dots y_{s_1, z_1}$ such that $s_d \geq 2$.

2.3.3. Associator equations.

2.3.3.a. The Lie algebras of $\pi_1^{dR}(M_{0,n}, can)$ recasted

To state the results below in their abstract algebraic context, it is convenient to introduce the following notations.

Definition 2.35. Let $n \in \mathbb{N}^*$.

- i) Let f_n , the topologically free Lie algebra over the generators x_1, \dots, x_n . In the case of f_2 we denote those generators by x, y .
- ii) Let t_n be the Lie algebra defined by generators t_{ij} , $i, j \in \{1, \dots, n\}$ with $i \neq j$; and relations :

$$t_{ij} = t_{ji}, [t_{ij}, t_{ik} + t_{jk}] = 0 \text{ for } \sharp\{i, j, k\} = 3 \text{ and } [t_{ij}, t_{kl}] = 0 \text{ for } \sharp\{i, j, k, l\} = 4.$$

Let \hat{t}_n be its completion for the degree.

Remark 2.36. Geometrically, the pro-nilpotent completion of f_2 is the Lie algebra of $\pi_1^{un, dR}(M_{0,4}, can)$. Below we identify (x, y) to (e_0, e_1) and thus $z = -x - y$ to e_{∞} . The pro-nilpotent completion of t_n is the Lie algebra of $\pi_1^{un, dR}(M_{0,n}, can)$.

2.3.3.b. Associator equations

In this paragraph we restrict to the case of $M_{0,4}$ and $M_{0,5}$, i.e. $Z = \{0, 1, \infty\}$. For our purposes, we will need "relative" analogues, not of the associator equations themselves, but of what we can call "pre-associator equations", which are identities between multiple polylogarithms. We will write them in §4.

Definition 2.37. [Dr] A series Φ is an associator is equivalent to the conjunction of

$$\Delta_{\mathfrak{m}}(\Phi) = \Phi \otimes \Phi, \quad \epsilon(\Phi) = 1$$

Φ is in the commutator subgroup of $\text{Spec}(\mathcal{H}_{\mathfrak{m}})(\mathbb{R})$

and the following set of equations, with, in this case, $m = 2i\pi$: they are called, respectively, *2-cycle*, *3-cycle* or *hexagon*, and *5-cycle* or *pentagon* equation.

$$\Phi(x, y)\Phi(y, x) = 1$$

$$e^{\frac{m}{2}x}\Phi(z, x)e^{\frac{m}{2}z}\Phi(y, z)e^{\frac{m}{2}y}\Phi(x, y) = 1$$

with $z = -x - y$, and

$$\Phi(t_{23}, t_{34})\Phi(t_{40}, t_{01})\Phi(t_{12}, t_{23})\Phi(t_{34}, t_{40})\Phi(t_{01}, t_{12}) = 1$$

Furusho has shown that the 5-cycle equation implies the 2-cycle and 3-cycle equations [F3].

2.3.4. Kashiwara-Vergne equations. We make a slight abuse of language, since the algebraic relations that we will describe here are not exactly the Kashiwara-Vergne equations. However, they are very close to Kashiwara-Vergne relations ; these are the relations obtained by Alekseev, Enriquez and Torossian as the main result of [AET]. In this paragraph we expose their results.

2.3.4.a. Automorphisms of free Lie algebras

The group of automorphisms of f_2 of the form $\mu : (x, y) \mapsto (UxU^{-1}, VyV^{-1})$ is denoted by $T \text{Aut}_2$ and is identified to $\exp(f_2)^2$ via $\mu \leftrightarrow (U, V)$.

Definition 2.38. For an associator Φ with constant m , let μ_Φ be the automorphism of f_2 defined by the couple $(U, V) = (\Phi(x, -x - y), \Phi(y, -x - y))$.

Definition 2.39. Let $\text{Ad} : \exp(\hat{t}_3) \rightarrow \exp(f_2)$ be the exponential of

$$ad : \quad t_{0i} \mapsto (x_j \mapsto [x_i, x_j]) \quad , \quad t_{ij} \text{ with } i, j > 0 \mapsto \begin{cases} x_i \mapsto [x_i, x_j] \\ x_j \mapsto [x_i, x_j] \\ x_k \mapsto 0 \text{ if } k \notin \{i, j\} \end{cases}$$

Definition 2.40. For a map $f : \{1, 2, 3\} \rightarrow \{1, 2\}$, there is a map $\exp(f_2)^2 \mapsto \exp(f_3)^2$ given by

$$(a_1(x_1, x_2), a_2(x_1, x_2)) \mapsto (a_{f(i)}(\sum_{k \in f^{-1}(1)} x_k, \sum_{k \in f^{-1}(2)} x_k)_{i=1,2,3} = (a_1, a_2)^{f^{-1}(1), f^{-1}(2)}$$

Remark 2.41. Under the isomorphism $f_2 \simeq \text{Lie } \pi_1^{un, dR}(M_{0,4}, can)$ given by $(x, y) \leftrightarrow (e_1, e_\infty)$, which amounts to choose 0 as a reference base point, μ_Φ maps

$$\begin{cases} e_0 \mapsto e_0 \\ e_1 \mapsto \Phi(e_0, e_1)^{-1} e^{me_1} \Phi(e_0, e_1) \\ e_\infty \mapsto e^{(m/2)e_0} \Phi(e_0, e_\infty)^{-1} e^{me_\infty} \Phi(e_0, e_\infty) e^{-(m/2)e_0} \end{cases}$$

2.3.4.b. The pseudo Kashiwara-Vergne relations

Definition 2.42. Let the pseudo Kashiwara-Vergne equation to be :

$$(10) \quad \text{Ad } \Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}$$

This equality appears in [AET] :

Theorem 2.43. (Alekseev-Enriquez-Torossian) If Φ is an associator, then μ_Φ satisfies (10).

This also has a particular consequence which also can be proved separately ([AET]) :

Proposition 2.44. With the same hypothesis, μ_Φ is special i.e. it satisfies

$$(11) \quad \mu_\Phi(\log e^x e^y) = x + y$$

2.3.5. Applications to multiple zeta values and hyperlogarithms.

2.3.5.a. Associator relations for multiple zeta values

The fact that Φ is an associator with $m = 2i\pi$ is a result of [?], §2. For the p -adic analogue, it is true for $\Phi_{p,-1}$ with $m = 0$, by the work of Ünver [U2].

By the work of Alekseev-Enriquez-Torossian, associators are solutions to the Kashiwara-Vergne problem [AET]

2.3.5.b. The double shuffle equations for multiple polylogarithms and multiple zeta values

2.3.5.b.a. The shuffle product for iterated integrals

Iterated integrals are integrals over simplices of \mathbb{R}^n , as in the formula (??). A product of two simplices of, respectively, \mathbb{R}^n and $\mathbb{R}^{n'}$ can be written as a disjoint union of simplices of $\mathbb{R}^{n+n'}$ up to sets of measure 0. This implies, for all words $w, w' \in \mathcal{H}_{\text{III}}(e_Z)$:

$$\begin{aligned} \text{Li}_w \text{Li}_{w'} &= \text{Li}_{w \text{III} w'} \\ \zeta_{\text{III}}(w) \zeta_{\text{III}}(w') &= \zeta_{\text{III}}(w \text{III} w') \end{aligned}$$

2.3.5.b.b. The series shuffle product for iterated sums

The expression of multiple zeta values as series, given at the beginning of the introduction, is a particular case of the formula for the series expansion at 0 of hyperlogarithms, given by

$$\text{Li}_{e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}}}(z) = \sum_{0 < n_1 < \dots < n_r < n} \sum \frac{\left(\frac{z_{i_2}}{z_{i_1}}\right)^{n_1} \dots \left(\frac{z_{i_{d+1}}}{z_{i_d}}\right)^{n_d} \left(\frac{z}{z_{d+1}}\right)^n}{n_1^{s_1} \dots n_d^{s_d}}$$

The series expansions of multiple zeta values and hyperlogarithms imply, for all $w \in \mathcal{H}_*(e_Z)$:

$$\begin{aligned} \zeta_*(w) \zeta_*(w') &= \zeta_*(w * w') \\ \text{Li}(w) \text{Li}(w') &= \text{Li}(w * w') \end{aligned}$$

2.3.5.b.c. Regularization of multiple zeta values and algebraic relations

The definition of multiple zeta values extends to the case where $s_d = 1$ into two different families of numbers $\zeta_*(s_d, \dots, s_1)$ and $\zeta_{\text{III}}(s_d, \dots, s_1)$ be the numbers indexed by $\Pi_{d \in \mathbb{N}^*}(\mathbb{N}^*)^d$, and obtained for $s_d = 1$ by regularizing multiple zeta values as, respectively, iterated sums and iterated integrals. There exists a simple formula relating ζ_* and ζ_{III} . The equation (6) remains true for $s_d = 1$ with ζ_{III} . See [C] for details.

3. COMPUTATIONAL SETTING

3.1. Choice of a base point at infinity. It is convenient to choose a base point at infinity as a reference. Branches of multiple polylogarithms are usually defined as the unique functions satisfying the KZ equation with a certain asymptotic condition at infinity. For the general case of $\overline{M}_{0,n}$, ([Br1], theorem 6.12.

On $\overline{M_{0,5}}$ we have to choose a point which is the intersection of two irreducible components of $\partial\overline{M_{0,5}}$.

In the cubic coordinates (c_1, c_2) introduced in §1.1.1, $(\mathbb{P}^1)^2 - M_{0,5}$ is a normal crossings at $(0,0)$, and the blow ups necessary to obtain $\overline{M_{0,5}}$ do not affect a neighbourhood of $(0,0)$. Thus it is convenient to take $O = (\vec{1}_0, \vec{1}_0)$. The same considerations and choice apply for the smooth compactification of $(\mathbb{P}^1 - Z)^2 - \Delta$,

On $M_{0,4}$, and more generally on $\mathbb{P}^1 - Z$, the natural choice is $\vec{1}_0$. It follows from classical arguments that

Proposition 3.1. i) There exists a unique holomorphic function L_1 on $(\mathbb{P}^1 - Z)(\mathbb{C})$ which is solution to ∇_{KZ} and such that $L_1(z)e^{-\log(z)e_0}$ is holomorphic at 0.
ii) There exists a unique holomorphic function L on $((\mathbb{P}^1 - Z^2 - \Delta)(\mathbb{C}))$ which is solution to ∇_{KZ} and such that $(L(c_1, c_2)e^{-\log(c_1)A \log(c_2)A'})$ is holomorphic at O .

In §4 we will also deal with other branches given by similar definitions at other base-points.

3.2. Taylor coefficients of hyperlogarithms.

Notation 3.2. We will denote the coefficients of power series $S =$

$$\sum_{(n_1, \dots, n_r) \in \mathbb{N}^r} a_{n_1, \dots, n_r} x_1^{n_1} \dots x_r^{n_r} \in \overline{\mathbb{Q}}[[x_1, \dots, x_r]] \text{ as follows : for all } (n_1, \dots, n_r) \in \mathbb{N}^r,$$

$$S[x_1^{n_1} \dots x_r^{n_r}] = a_{n_1, \dots, n_r}$$

3.2.1. Finite iterated sums and Taylor coefficients of multiple polylogarithms .

3.2.1.a. $\mathbb{P}^1 - Z$

Let us recall that we take the notation $Z = \{0, z_1, \dots, z_r, 1, \infty\}$ and $z_0 = 0$ and $z_{r+1} = 1$ for a finite subset of $\mathbb{P}^1(\overline{\mathbb{Q}})$. The sequences $(\text{Li } \mathcal{T})_{O, \text{prime}}[\tilde{w}]$ of the introduction are related to Taylor coefficients of hyperlogarithms in two slightly different ways.

Let an index of multiple harmonic sums $\tilde{w} = \begin{pmatrix} z_{i_{d+1}}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix}$. We associate to it

two different indices of hyperlogarithms : $w_l = \begin{pmatrix} z_{i_{d+1}}, \dots, z_{i_1} \\ l, s_d, \dots, s_1 \end{pmatrix}$, for $l \in \mathbb{N}^*$, and also

$\partial w = \begin{pmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix}$. We have, for z on a neighbourhood of 0 in \mathbb{C} :

$$(12) \quad \text{Li}(w)(z) = \sum_{0 < n_1 < \dots < n_d < n} \frac{(z_{i_2}/z_{i_1})^{n_1} \dots (z_{i_{d+1}}/z_{i_d})^{n_{i_d}} (z/z_{i_{d+1}})^n}{n_1^{s_1} \dots n_{i_d}^{s_d} n^l}$$

Whence the two different expressions :

Fact 3.3. We have :

$$(13) \quad (\text{Li } \mathcal{T})_O[\tilde{w}] = \left(n^{l+s_d+\dots+s_1} \text{Li}[w_l][z^n] \right)_{n \in \mathbb{N}} = \left(n^{s_d+\dots+s_1} \sum_{0 < m < n} z_{i_{d+1}}^{m-n} \text{Li}[\partial w][z^m] \right)_{n \in \mathbb{N}}$$

Note that the first equality is true for all $l \in \mathbb{N}^*$.

3.2.1.b. $X_Z^2 - \Delta_Z$

Here, we give an idea why we obtain the same numbers by considering similar sequences of Taylor coefficients in higher dimensions.

Let $w = \begin{pmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix}$, $w' = \begin{pmatrix} z_{j_{d'}}, \dots, z_{j_1} \\ t_{d'}, \dots, t_1 \end{pmatrix}$ be indices of hyperlogarithms. We consider an iterated integral on $\overline{M_{0,5}}(\mathbb{C})$, regular at O :

Lemma 3.4. For $(y_1, y_2) \in \overline{M_{0,5}}(\mathbb{C})$ on a neighbourhood of O as in §2.1 we have :

$$(14) \quad \int_O^{(y_1, y_2)} \omega_0^{t_{d'}-1}(y_2) \omega_{z_{j_{d'}}}(y_2) \dots \omega_0^{t_1-1}(y_2) \omega_{z_{j_1}}(y_2) \\ \omega_0^{s_d-1}(y_1 y_2) \omega_{z_{i_d}}(y_1 y_2) \dots \omega_0^{s_1-1}(y_1 y_2) \omega_{z_{i_1}}(y_1 y_2) \\ = \sum_{0 < n_1 < \dots < n_d < m_1 < \dots < m_{d'}} \frac{(z_{i_2}/z_{i_1})^{n_1} \dots (y_1/z_{i_d})^n (z_{j_2}/z_{j_1})^{n_1} \dots (y_2/z_{j_{d'}})^m}{n_1^{s_1} \dots n_d^{s_d} m_1^{t_1} \dots m_{d'}^{t_{d'}-1}} m^{t_{d'}} \Big)$$

This equality does not yields new numbers. The Taylor coefficients on this formula are, essentially, multiple harmonic sums with indices equipped with an additional data of marking of one of the s_i 's.

3.2.2. Product of two hyperlogarithms . Let indices of hyperlogarithms, $w = \begin{pmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix}$, $w' = \begin{pmatrix} z_{j_{d'}}, \dots, z_{j_1} \\ t_{d'}, \dots, t_1 \end{pmatrix}$ in $\mathcal{H}_{\text{in}}(e_Z)$.

Let us write the product of the associated hyperlogarithms on the level of Taylor coefficients :

$$(15) \quad (\text{Li}_{w_1} \text{Li}_{w_2})[z^n] = \sum_{0 < m < n} \text{Li}_{w_1}[z^m] \text{Li}_{w_2}[z^{n-m}]$$

The right hand side is not intrinsically expressed in terms of multiple harmonic sums. This leads us to define :

Definition 3.5. Let the (associative, commutative) product \times on the vector space generated by $\text{Li } \mathcal{T}_O$ in $\overline{\mathbb{Q}}^{\mathbb{N}}$ be defined as :

$$(\text{Li } \mathcal{T}_O[\tilde{w}_1] \times (\text{Li } \mathcal{T}_O[\tilde{w}_2]) = \left(n^{\sum s_i + \sum t_j} (\text{Li}_{w_1} \text{Li}_{w_2})[z^n] \right)_{n \in \mathbb{N}^*}$$

and $(\text{Li } \mathcal{T}_{O, \text{prime}}[\tilde{w}_1] \times (\text{Li } \mathcal{T}_{O, \text{prime}}[\tilde{w}_2])$ be their subsequences indexed by powers of p , viewed in $\left(\prod_{p \text{ prime}} \overline{\mathbb{Q}_p} \right)^{\mathbb{N}^*}$.

In the case of prime multiple harmonic sums, the product $(\text{Li } \mathcal{T}_{O, \text{prime}}[\tilde{w}_1] \times (\text{Li } \mathcal{T}_{O, \text{prime}}[\tilde{w}_2])$ has an intrinsic expression in terms of prime multiple harmonic sums, which involves an absolutely convergent p -adic infinite sum.

Fact 3.6. We have, for all words :

$$(16) \quad (\text{Li } \mathcal{T})_{O, \text{prime}}[\tilde{w}_1] \times (\text{Li } \mathcal{T})_{O, \text{prime}}[\tilde{w}_2] = \sum_{l_1, \dots, l_d \in \mathbb{N}} \prod_{i=1}^d \binom{-t'_i}{l_i} (-1)^{t'_i} (\text{Li } \mathcal{T})_{O, \text{prime}} \left[\begin{matrix} y_1, \dots, y_d, z_d, \dots, z_1 \\ t'_1 + l_1, \dots, t'_{d-1} + l_{d-1}, t'_d + s_d + l_{d'}, s_{d-1}, \dots, s_1 \end{matrix} \right]$$

Indeed, given a prime number p , the positive powers of p are the integers $n \in \mathbb{N}^*$ satisfying the implication $0 < m < n \Rightarrow v_p(m) < v_p(n)$, which enable to write certain p -adic series expansions.

Proof. Starting from 15 $\text{Li}_{w_2}[z^{n-m}]$ is equal to :

$$= \sum_{0 < n_1 < \dots < n_d = n-m} \frac{(z_2/z_1)^{n_1} \dots (1/z_d)^{n_d}}{n_1^{s_1} \dots n_d^{s_d}} = \sum_{m=n'_d < \dots < n'_1 < n} \frac{(z_2/z_1)^{n-n'_1} \dots (1/z_d)^{n-n'_d}}{(n-n'_d)^{s_d} \dots (n-n'_1)^{s_1}}$$

By expanding the factors of the form $\frac{1}{(m-p^k)^s}$ into p -adic series $m^{-s} \sum_{l \geq 0} \binom{-s}{l} \left(\frac{p^k}{m}\right)^l \in \mathbb{Z}_p$, we have :

$$(17) \quad \text{Li}_{w_1} \text{Li}_{w_2}[z^{p^k}] = \left(\sum_{l_1, \dots, l_d \in \mathbb{N}} \times \prod_{i=1}^d \binom{-t'_i}{l_i} (-1)^{t'_i} \right) \frac{1}{y_1^{p^k}} \sum_{\substack{0 < n_1 < \dots < n_{d-1} < n_d = n \\ = m_d < m_{d-1} < \dots < m_1 < p^k}} \frac{(z_2/z_1)^{n_1} \dots (1/z_d)^n y_{d'}^n (y_{d'-1}/y_{d'})^{m_{d-1}} \dots (y_2/y_1)^{n_1}}{n_1^{s_1} \dots m_d^{s_d+t_{d'}+l_{d'}} m_{d-1}^{s_{d-1}+l_{d-1}} \dots m_1^{s'_1+l_1}}$$

□

The formulas of this paragraph explain why we defined $\text{Li } \mathcal{T}$ using the multiplication by (upper bound^{weight} ; as for the reason why we decided to define prime multiple harmonic sums as living in $\left(\prod_{p \text{ prime}} \overline{\mathbb{Q}_p} \right)^{\mathbb{N}^*}$, although another reason was their expression in terms of their images by Frobenius, stated in the introduction.

3.3. Images by Frobenius and analogues.

3.3.1. *From hyperlogarithms $\Phi_{0z}[w]$ to the analogues of the images by Frobenius of prime multiple harmonic sums $(\Phi_{0z}^{-1} e_1 \Phi_{0z})[\frac{1}{1-\Lambda e_0} e_1 w]$.*

3.3.1.a. Introduction

We now define maps of "reindexation" of iterated integrals - that we will denote by Σ as a reference to the infinite summations of p -adic multiple zeta values.

They can be defined maps from the algebras of functions on the Betti or de Rham fundamental group to itself, or its weight-adic completion.

Thus, they are either maps modifying the differential forms (Σ_ω) or the path of integration (Σ_γ) .

In the next parts, from §4, to §7, we will show how we can "transfer" the usual algebraic properties of multiple zeta values or hyperlogarithms onto analogous properties for the prime multiple harmonic sum motive, along these maps.

Aside from the definition, we give heuristic reasons why it is reasonable to hope that those transfers of algebraic properties are possible.

3.3.1.b. From Φ_{0z} to $\Phi_{0z}^{-1}e_z\Phi_{0z}$

We fix $z \in Z - \{0, \infty\}$, and a non-zero rational tangent vector \vec{v} at z to \mathbb{P}^1 .

Definition 3.7. let the map

$$\begin{aligned} \Sigma_\gamma : \pi_1^{dR,un}(X_Z, \vec{1}_0, \vec{v}_z)(\mathbb{C}) &\rightarrow \text{Lie}^\vee \pi_1^{dR,un}(X_Z, \vec{1}_0, \vec{1}_0) \otimes_K \mathbb{C} \\ f &\mapsto f^{-1}e_z f \end{aligned}$$

The reason why, for all f , $f^{-1}e_z f$ is a Lie series is that it is conjugation of the element e_z primitive for Δ_{III} by an element f , grouplike for Δ_{III} ; it is thus primitive for Δ_{III} .

The map Σ_γ is a sort of "symmetrization" of the path of integration. Its image involves to integrate both on a path from 0 to z , and from z to 0. Precisions on the topological meaning of this map will be given in 7.

Fact 3.8. Σ_γ , restricted to the subgroup of series f which vanish in weight one, i.e. such that $f[e_{z_i}] = 0$ for all i , is injective.

Proof. Let $u \in R\langle\langle e_0, e_{z_0}, \dots, e_{z_r} \rangle\rangle$. Then u commutes to, for example, e_1 (resp. commutes to e_1 and is a grouplike series) if and only if $u \in R\langle\langle e_1 \rangle\rangle$ (resp. is in $\exp(Re_1)$). Indeed, let w a word not of the form e_1^n , $n \geq 1$. We have $u[w] = (ue_1)[we_1] = (e_1u)[we_1] = u(\partial_{e_1}(w)e_1)$. Because of the hypothesis on w , this shows $u[w] = 0$ by induction on the index of nilpotence of w relatively to ∂_{e_1} . \square

Fact 3.9. The \mathbb{Q} -algebras of coefficients Φ and $\Phi_{0z}^{-1}e_z\Phi_{0z}$ are the same : this is stated precisely and explicitly as a lemma in our paper on associator equations and the depth filtration (in the case where $Z = \{0, 1, \infty\}$, but the general case is the same) : [J6].

Heuristically, those two facts show that Σ_γ does not lose information. Moreover, the fact that the images by Σ_γ are Lie series suggests already a way to obtain a shuffle equation (modulo products).

There is a third indication that applying Σ_γ yields nice variants of the known families of relations. It is that the Ihara action can be pushed forward by sym into a "symmetric Ihara action", which is given by a nice formula, which is actually much simpler combinatorially than the usual one for the Ihara action : it is given by $g \circ_{\text{sym}} f = f(e_0, g)$. We will discuss it in part III.

3.3.1.c. Maps from w to an infinite sum of words

We take notations for maps of the form $w \mapsto \frac{1}{1-\Lambda e_0} e_{z_i} w$.

For maps which are adapted to the Lie algebra of $\pi_1^{\text{dR}}(X_Z; \vec{1}_0, \vec{1}_0)$, which will be signaled by the exponent Lie , we will use for convenience the different notations. The multiplication by $\frac{1}{\Lambda}$ corresponds to the shift of 1 in the weight existing between the coefficients of Φ_{0z} and the ones of $\Phi_{0z}^{-1}e_z\Phi_{0z}$.

Definition 3.10. (maps adapted to $\pi_1^{un,dR}(\mathbb{P}^1 - Z; \vec{v}_z, \vec{1}_0)$)

i) Let

$$(\Sigma_\omega)_\mathfrak{m} : \begin{array}{c} \mathcal{H}_\mathfrak{m}(Z) \longrightarrow \widehat{\mathcal{H}_\mathfrak{m}(Z)} \\ w \longmapsto w \frac{1}{1-e_0} \end{array}$$

ii) The map $\iota_\mathfrak{m}$ induces

$$(\Sigma_\omega)_* : \begin{array}{c} \mathcal{H}_*(Z) \hookrightarrow \mathcal{H}_\mathfrak{m}(Z) \xrightarrow{\iota} \widehat{\mathcal{H}_\mathfrak{m}(Z)} \xrightarrow{pr} \widehat{\mathcal{H}_*(Z)} \\ w(e_0, e_1) \mapsto w\left(\frac{1}{1+e_0}e_0, \frac{1}{1+e_0}e_{z_1}, \dots, \frac{1}{1+e_0}e_{z_r}\right) \end{array}$$

where the quotient pr corresponds to the relations arising from (9), satisfied by points f of $\pi_1^{dR}(X; \vec{v}_z, \vec{1}_0)$ with $f[e_0] = 0$.

We have, for all such f , $f[\iota(w)] = f[\iota_*(w)]$ for $w \in \mathcal{H}_*$. Then ι_* is the unique morphism of concatenation algebras satisfying : $\iota_* : y_s \mapsto \left(\frac{1}{1+e_0}\right)^s y_s = \sum_{l \geq 0} (-1)^l \binom{l+s-1}{s-1} y_{s+l}$.

Definition 3.11. (maps adapted to $\text{Lie } \pi_1^{dR}(X_Z; \vec{v}_z, \vec{1}_0)$)

Let, for all i :

$$\Sigma_{\omega, \text{Lie}}(z_i) : \mathcal{H}_\mathfrak{m}(Z) \longrightarrow \widehat{\mathcal{H}_\mathfrak{m}(Z)}, w \mapsto \frac{1}{1-e_0} e_{z_i} w$$

We have, for all point f of $\pi_1^{dR}(X_Z; \vec{v}_z, \vec{1}_0)$ satisfying $f[e_0] = 0$, and $w \in \mathcal{H}_*$, that $(f^{-1}e_z f)[j^{\text{Lie}}(w)] = (f^{\text{inv}} \circ \iota_*)f[w]$.

where

Notation 3.12. $\text{inv} : \mathcal{H}_*(Z) \rightarrow \mathcal{H}_*(Z)$ is the unique anti-morphism of concatenation algebras satisfying $\text{inv}(y_n) = (-1)^n y_n$ and $\phi \mapsto \phi^{\text{inv}}$ is its dual.

3.3.1.d. The weight homogeneity

The maps of reindexation of the differential forms will involve different values of the weight. We will modify them by introducing a formal variable reflecting the weight of the corresponding term.

In the language of the motivic Galois group of the category of mixed Tate motives over \mathbb{Z} , the conjectural weight homogeneity of the relations is expressed by saying that, the motivic multiple zeta values are acted upon by \mathbb{G}_m , a subgroup of the motivic Galois group, by $\lambda \mapsto$ (multiplication by λ^{weight}). This amounts to say that the weight homogeneity of algebraic is true on the level on motivic multiple zeta values, and then, by the usual conjectures on motivic multiple zeta values, is conjecturally true on the level of multiple zeta values.

Notation 3.13. We will view and denote the action of \mathbb{G}_m , as a subgroup of the Galois group G^ω of the Tannakian category of mixed Tate motives over a number field, on $\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{1}_0)$, as follows :

$$\Lambda : \mathcal{O}(\pi_1^{dR}(X_Z; \vec{v}_z, \vec{1}_0)) \hookrightarrow \mathcal{O}(\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{1}_0))[\Lambda]$$

, $w \mapsto \Lambda.w$, where Λ is a formal variable. It is the map defined by $w \mapsto \Lambda^{\text{weight}(w)}w$. It associates to a word its orbit under the action of \mathbb{G}_m .

This is a linear map between graded algebras which induces a map on their respective completions

$$\mathcal{O}(\widehat{\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{I}_0)}) \hookrightarrow \mathcal{O}(\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{I}_0))[[\Lambda]]$$

and it fits into a commutative diagram.

$$(18) \quad \begin{array}{ccc} \mathcal{O}(\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{I}_0)) & \hookrightarrow & \mathcal{O}(\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{I}_0))[[\Lambda]] \\ \downarrow & & \downarrow \\ \mathcal{O}(\widehat{\pi_1^{un,dR}(X; \vec{v}_z, \vec{I}_0)}) & \hookrightarrow & \mathcal{O}(\pi_1^{un,dR}(X_Z; \vec{v}_z, \vec{I}_0))[[\Lambda]] \end{array}$$

The series shuffle Hopf algebra $\mathcal{H}_*(Z)$ is also graded by the weight, can be viewed as a vector subspace of \mathcal{H}_{III} , and the analogous diagram as (22) holds.

Notation 3.14. (inverse of the motivic Galois action)

Λ^{-1} refers below to the inverse of the action of \mathbb{G}_m : it multiplies words by $\Lambda^{-\text{weight}}$.

Notation 3.15. (composition and conjugation by the action of \mathbb{G}_m).

These notations will appear throughout the paper. For s a linear map from the series of integral shuffle Hopf algebra to its completion, we denote by

$$\begin{aligned} \Lambda s &= \Lambda \circ s \\ \Lambda s \Lambda^{-1} &= \Lambda \circ s \circ \Lambda^{-1} \end{aligned}$$

For example :

$$\begin{aligned} \Lambda \Sigma_{\omega} \Lambda^{-1} : w &\mapsto w \frac{1}{1 - \Lambda e_0} \\ \Lambda \Sigma_{\omega, \text{Lie}}(z_i) \Lambda^{-1} : w &\mapsto \frac{\Lambda}{1 - \Lambda e_0} e_{z_i} w \end{aligned}$$

We note that, for $s \geq 0$, and $w = y_s w'$:

$$(19) \quad \frac{1}{(s-1)!} \left(\frac{\partial}{\partial \Lambda} \right)^{s-1} (\Lambda \Sigma_{\omega, \text{Lie}} \Lambda^{-1})(w') = (\Lambda \Sigma_*(y_s) \cdot w$$

Notation 3.16. We will sometimes use :

$$(\Sigma_{\text{III}}^{\text{sym}})_{\Lambda, \Lambda'}(z_i) : \mathcal{H}_{\text{III}} \rightarrow \mathcal{H}_{\text{III}}[[U, T]], w \mapsto \frac{1}{1 - \Lambda e_0} e_{z_i} w \frac{1}{1 - \Lambda' e_0}$$

4. EQUATIONS OF DOUBLE SHUFFLE TYPE

4.1. Statement. We state the theorem on double shuffle relations.

We recall that all relations among $(\text{Li } \mathcal{T})_{O, \text{prime}}^{\hat{M}}$ are automatically true for $(\text{Li } \mathcal{T})_{O, \text{prime}}^{\mathbb{C}[[\Lambda]]}$ and $(\text{Li } \mathcal{T})_{O, \text{prime}}^{\mathbb{C}_p[[\Lambda]]}$; moreover, the indices of multiple harmonic sums are sequences of the

form $\begin{pmatrix} z_{i_{d+1}}, \dots, z_1 \\ s_d, \dots, s_1 \end{pmatrix}$, $s_j \in \mathbb{N}^*$, $i_{j'} \in \{1, \dots, r\}$.

Let also \times be the product defined in 3.2.2.

Theorem 1 : equations of double shuffle type

• **Taylor coefficients**

For all w, w' , indices of multiple harmonic sums, we have the following equalities, implied by relations among hyperlogarithms via taking Taylor coefficients :

i) Series shuffle relation :

$$(\text{Li } \mathcal{T})_O[w * w'] = (\text{Li } \mathcal{T})_O[w] \cdot (\text{Li } \mathcal{T})_O[w']$$

$$(\text{Li } \mathcal{T})_{O,\text{prime}}[w * w'] = (\text{Li } \mathcal{T})_{O,\text{prime}}[w] \cdot (\text{Li } \mathcal{T})_{O,\text{prime}}[w']$$

ii) Integral shuffle relation :

$$(\text{Li } \mathcal{T})_O[w \boxtimes w'] = (\text{Li } \mathcal{T})_O[w] \times (\text{Li } \mathcal{T})_O[w']$$

$$(\text{Li } \mathcal{T})_{O,\text{prime}}[w \boxtimes w'] = (\text{Li } \mathcal{T})_{O,\text{prime}}[(\Lambda(\Sigma_\omega)_* \Lambda^{-1} \circ \text{inv})(w')w]$$

• **images by Frobenius and analogues**

i) Series shuffle relation : for all indices of multiple harmonic sums :

$$(\text{Li } \mathcal{T})_{O,\text{prime}}^{\widehat{\mathcal{M}}}[w * w'] = (\text{Li } \mathcal{T})_{O,\text{prime}}^{\widehat{\mathcal{M}}}[w] \cdot (\text{Li } \mathcal{T})_{O,\text{prime}}^{\widehat{\mathcal{M}}}[w'] \pmod{\zeta(2)}$$

ii) Integral shuffle relation : for all indices of multiple harmonic sums we have, via the

correspondence $\left(\begin{smallmatrix} z_{i_{d+1}}, \dots, z_1 \\ s_d, \dots, s_1 \end{smallmatrix} \right) \leftrightarrow e_{z_{i_{d+1}}} e_0^{s_d-1} e_{z_{i_d}} \dots e_0^{s_1-1} e_{z_{i_1}} :$

$$(\text{Li } \mathcal{T})_{O,\text{prime}}^{\widehat{\mathcal{M}}}[e_{z_j}(e_0^{s-1} e_{z_i} w \boxtimes w')] = (\text{Li } \mathcal{T})_{O,\text{prime}}^{\widehat{\mathcal{M}}}[e_{z_i}(w \boxtimes \frac{1}{1 - \Lambda e_0} e_0^{s-1} e_{z_j} w')]$$

The equivalence between the two versions of the integral shuffle relation is explained in §4.3.

Corollary 4.1. The shuffle equation implies for both objects a "symmetry" relation :

$$(20) \quad (\text{Li } \mathcal{T})_{O,\text{prime}}[(\Sigma_\omega)_* \text{inv})(w')w] = (\text{Li } \mathcal{T})_{O,\text{prime}}[(\Sigma_\omega)_* \text{inv})(w)w']$$

By taking $w' = \emptyset$ we obtain :

$$(21) \quad (\text{Li } \mathcal{T})_{O,\text{prime}}[(\Sigma_\omega)_* \text{inv})(w)] = (\text{Li } \mathcal{T})_{O,\text{prime}}[w]$$

Remark 4.2. The analogy between the two setting applies also to the comparison of regularizations of §2.2.3 : integral regularization and series regularization.

- Taylor coefficients : they are defined without reference to a regularization.

- Images by Frobenius and analogues : the two regularizations coincide modulo $\zeta(2)$, as we explain in §4.3. The usual Φ_{0z} is defined through integral regularization.

Remark 4.3. In [R], Rosen has stated an "asymptotic reflexion theorem" for certain prime multiple harmonic sums, which is a particular case of the last symmetry equation above.

4.2. Proof : Taylor coefficients. The proof of the double shuffle relations in the case i) of Taylor coefficients are straightforward, having defined the setting of §3.2.

Proof. (of i) of the theorem) For the series shuffle equation, there is nothing to prove, since the usual series shuffle equation follows precisely from the series expansion at 0 of hyperlogarithms.

Consider the integral shuffle equation for hyperlogarithms : for all words $w, w' \in e_Z$,

$$\text{Li}_{w \boxtimes w'} = \text{Li}_w \text{Li}_{w'}$$

For words w, w' such that $\tilde{\partial}_{e_1}(w) = \tilde{\partial}_{e_1}(w') = 0$, we translate this equation on the sum of coefficients $\sum_{0 < m < n} [z^m]$ at 0, and we multiply the equality by $n^{\text{weight}(w) + \text{weight}(w')}$.

Both sides of the equality are expressed in terms of multiple harmonic sums by Fact 3.3. When n is a power of a prime number, the right hand side is expressed as a p -adic infinite sum of prime multiple harmonic sums by Fact 3.6. \square

4.3. Proofs : images by Frobenius and analogues.

4.3.1. *Series shuffle relation.* We first prove the theorem in the special case of $\mathbb{P}^1 - \{0, 1, \infty\}$.

Lemma 4.4. i) $\text{inv} : \mathcal{H}_*(e_{\{0,1,\infty\}}) \rightarrow \mathcal{H}_*(e_{\{0,1,\infty\}})$ is a anti-morphism of series shuffle algebras.

ii) $(\Sigma_\omega)_*$ is a morphism of series shuffle algebras

Proof. i) Clear. ii) The dual of $(\Sigma_\omega)_*$ is the concatenation algebra morphism ι_*^\vee defined by

$$y_s \mapsto \Lambda^s \sum_{l=0}^{s-1} \Lambda^l \binom{s-1}{l} (-1)^{s-l} y_{s-l} = \Lambda^s \sum_{l=1}^s \Lambda^{s-l} (-1)^l y_l \binom{s-1}{s-l}$$

We have $(\iota^\vee \otimes \iota^\vee) \Delta_*(y_s) = 1 \otimes \iota^\vee(y_s) + \iota^\vee(y_s) \otimes 1 + \sum_{k=1}^{s-1} \iota^\vee(y_k) \otimes \iota^\vee(y_{s-k})$; the third term of this sum is

$$\begin{aligned} & \Lambda^s \sum_{k=1}^{s-1} \left(\sum_{l=1}^k \Lambda^{k-l} \binom{k-1}{k-l} (-1)^l y_l \right) \otimes \left(\sum_{l'=1}^{s-k} \Lambda^{s-k-l} \binom{s-k-1}{s-k-l'} (-1)^{l'} y_{l'} \right) \\ &= \Lambda^s \sum_{L=2}^s \Lambda^{s-L} (-1)^{s-L} \left(\sum_{\substack{l+l'=L \\ l, l' \geq 1}} y_l \otimes y_{l'} \right) \sum_{l \leq k \leq s-L+l} \binom{k-1}{k-l} \binom{s-k-1}{s-k-l'} \end{aligned}$$

For all l, l' such that $l + l' = L$,

$$\sum_{l \leq k \leq s-L+l} \binom{k-1}{k-l} \binom{s-k-1}{s-k-l'} = \sum_{k'=0}^{s-L} \binom{k'+l-1}{k'} \binom{s-L-k'+l'-1}{s-L-k'} = \binom{s-L+L-1}{s-L}$$

This gives the result. \square

Let the dual - in the sense of the duality between $\pi_1^{dR}(X_Z, \vec{1}_0, -\vec{1}_0)$ and its algebra of functions - of $\Sigma_\gamma : \Phi_{01} \mapsto \Phi_{01}^{-1} e_1 \Phi_{01}$, restricted to $\mathcal{H}_*(e_Z)$:

$$\begin{aligned} & \mathcal{H}_* \rightarrow \widehat{\mathcal{H}}_* \otimes \mathcal{H}_* \\ \Sigma_\gamma^\vee : \begin{pmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{pmatrix} & \mapsto \sum_{1 \leq k \leq d | z_{i_k}=1} \begin{pmatrix} z_{k+1}, \dots, z_{d+1} \\ s_k, \dots, s_d \end{pmatrix} \otimes \begin{pmatrix} z_{k-1}, \dots, z_1 \\ s_{k-1}, \dots, s_1 \end{pmatrix} \end{aligned}$$

We consider

$$i = \Lambda(\Sigma_\omega)_* \Lambda^{-1} \circ \Sigma_\gamma^\vee$$

By its definition, the map i satisfies : for all $w \in \mathcal{H}_*(e_Z)$, $(\Phi_{01}^{-1} e_1 \Phi_{01})[\frac{1}{1-\Lambda e_0} e_1 w] = \text{mult} \circ (\Phi \otimes \Phi)(i(w))$.

Lemma 4.5. The following diagram is commutative :

$$(22) \quad \begin{array}{ccc} \mathcal{H}_* \otimes \mathcal{H}_* & \xrightarrow{*} & \mathcal{H}_* \\ \downarrow i \otimes i & & \downarrow i \\ (\widehat{\mathcal{H}}_* \otimes \mathcal{H}_*) \otimes (\widehat{\mathcal{H}}_* \otimes \mathcal{H}_*) & \xrightarrow{(* \otimes *) \circ (1324)} & (\widehat{\mathcal{H}}_* \otimes \mathcal{H}_*) \end{array}$$

where (1324) : $w_1 \otimes w_2 \otimes w_3 \otimes w_4 \mapsto w_1 \otimes w_3 \otimes w_2 \otimes w_4$

Proof. Similar with the previous lemma. \square

4.3.2. Integral shuffle relations.

Lemma 4.6. i) For all $w, w' \in \mathcal{H}_*(Z)$:

$$j_{\Lambda}^{\text{Lie}}(z_i)(w) \mathfrak{m} j_{\Lambda'}^{\text{Lie}}(z_j)(w') = j_{\Lambda+\Lambda'}^{\text{Lie}}(z_i) \left(w \mathfrak{m} j_{\Lambda'}^{\text{Lie}}(z_j)(w') \right) + j_{\Lambda+\Lambda'}^{\text{Lie}}(z_j) \left(j_{\Lambda}^{\text{Lie}}(z_i)(w) \mathfrak{m} w' \right)$$

Proof. We use that x is equal to $\sum_{i=0}^r e_{z_i} \partial_{e_{z_i}}(x)$, and we compute, for all $i = 0, \dots, r$,

$$\partial_{e_i} \left(j_T^{\text{Lie}}(z_i)(w) \mathfrak{m} j_U^{\text{Lie}}(z_j)(w') \right)$$

using that ∂_{e_i} is a derivation for \mathfrak{m} . This gives the result. \square

Remark 4.7. Applying $(\frac{\partial}{\partial \Lambda})^{s-1} (\frac{\partial}{\partial U})^{s'-1}$ to the previous lemma and using (19) we can write a formula for $\iota_{*,T} \mathfrak{m} \iota_{*,U}$.

Proposition 4.8. We have :

(23)

$$-j_T^{\text{Lie}} \left((e_0^{s-1} e_1 w) \mathfrak{m} w' - w \mathfrak{m} (\Sigma_{\omega})_*(e_0^{s-1} e_1 w') \right) = \sum_{k=0}^{s-1} (e_0^k e_1 w) \mathfrak{m} ((-1)^{s-k} (\Sigma_{\omega})_*(e_0^{s-1-k} e_1 w'))$$

Proof. Let r be the right-hand side. It suffices to show the following equalities :

if $i \neq j$, $\partial_{e_{z_i}}(r) = w \mathfrak{m} (-1)^s (\Sigma_{\omega})_*(e_0^{s-1} e_{z_j} w')$ and $\partial_{e_{z_j}}(r) = -(e_0^{s-1} e_{z_i} w) \mathfrak{m} w'$;

if $i = j$, $\partial_{e_{z_i}}(r) = w \mathfrak{m} (-1)^s (\Sigma_{\omega})_*(e_0^{s-1} e_{z_i} w') - (e_0^{s-1} e_{z_i} w) \mathfrak{m} w'$;

$\partial_{e_0}(r) = \Lambda r$.

All equalities are clear except for the last one. For the last one, we have :

$$(24) \quad \partial_{e_0}(r) = \sum_{k=1}^{s-1} (e_0^{k-1} e_{z_i} w) \mathfrak{m} \frac{1}{(\Lambda e_0 - 1)^{s-k}} e_0^{s-1-k} e_{z_j} w' \\ + \sum_{k=0}^{s-2} (e_0^k e_{z_i} w) \mathfrak{m} \frac{1}{(\Lambda e_0 - 1)^{s-k}} e_0^{s-2-k} e_{z_j} w' + (e_0^{s-1} e_{z_i} w) \mathfrak{m} \frac{\Lambda}{\Lambda e_0 - 1} e_{z_j} w'$$

The sum of the two first terms of the right hand side of (24) equals

$$\sum_{k=0}^{s-2} (e_0^k e_{z_i} w) \mathfrak{m} \left[1 + \frac{1}{\Lambda e_0 - 1} \right] \frac{1}{(\Lambda e_0 - 1)^{s-1-k}} e_0^{s-2-k} e_{z_j} w' \\ = \Lambda \sum_{k=0}^{s-2} (e_0^k e_{z_i} w) \mathfrak{m} \frac{1}{(\Lambda e_0 - 1)^{s-k}} e_0^{s-1-k} e_{z_j} w'$$

This and the third term of (24) are, respectively, the $0 \leq k \leq s-2$ terms and the $k = s-1$ term of Λr . \square

Remark 4.9. The $\Lambda = 0$ case is

$$\sum_{k=0}^{s-1} (e_0^k e_{z_i} w) \mathfrak{m} ((-1)^{s-1-k} e_0^{s-1-k} e_{z_j} w') = e_{z_j} ((e_0^{s-1} e_{z_i} w) \mathfrak{m} w') + (-1)^{s-1} e_{z_i} (w \mathfrak{m} (e_0^{s-1} e_{z_j} w'))$$

In this case, the same proof is that $\partial_{e_0}(r)$ gives directly a telescopic sum which vanishes.

If $h : \mathcal{H}_{\text{III}} \rightarrow R$ satisfies the shuffle equation modulo products, then we have in particular, for all $w, w' \in W$ and $s \in \mathbb{N}^*$:

$$h\left[\sum_{k=0}^{s-1} (e_0^k e_1 w) \text{III}((-1)^{s-k} (\Sigma_\omega)_*(e_0^{s-1-k} e_1) w')\right] = 0$$

Lemma 4.10. Let F a function $\ker \tilde{\partial}_{e_0} \rightarrow R$ and \tilde{i} , a function $\ker \tilde{\partial}_{e_0} \rightarrow \ker \tilde{\partial}_{e_0}[[\Lambda]]$ satisfying, for all $a, b \in \ker \tilde{\partial}_{e_0}$, $\tilde{i}(ab) = \tilde{i}(b)\tilde{i}(a)$.

We have an equivalence between :

- i) $\forall s \in \mathbb{N}^*, i = 1, \dots, r, \forall w, w' \in \tilde{\partial}_{e_0}, F((e_0^{s-1} e_{z_i} w) \text{III} w') = F(w \text{III} (\tilde{i}(e_0^{s-1} e_{z_i}) w'))$
- ii) $\forall u, w, w' \in \tilde{\partial}_{e_0}, F((uw) \text{III} w') = F(w \text{III} (\tilde{i}(u) w'))$
- iii) $\forall w, w' \in \tilde{\partial}_{e_0}, F(w \text{III} w') = F(\tilde{i}(w) w')$.

Proof. i) \Rightarrow ii) : we write u as a concatenation of words of the form $e_0^{s_i-1} e_1$ and we iterate iii).

ii) \Rightarrow iii) : we take $w = \emptyset$.

iii) \Rightarrow i) : we apply iii) to each member of i). □

The combination of all the lemmas of this paragraph gives the part for images by Frobenius of prime multiple harmonic sums and their analogues of the theorem 1.

4.3.3. Relation between the two regularizations. For simplicity we take the case of $\mathbb{P}^1 - \{0, 1, \infty\}$, but the general case is similar. Let (f_*, f_{III}) be a solution of the double shuffle equations.

Lemma 4.11. We have $f_*^{\text{inv}} f_* \equiv \widetilde{f_{\text{III}}}^{\text{inv}} \widetilde{f_{\text{III}}} \pmod{f[y_2]}$.

Proof. Let $\widetilde{f_{\text{III}}}$ be the function $\mathcal{H}_* \rightarrow R$ equal to the restriction of f to $\mathbb{Q}\langle e_0, e_1 \rangle e_1$ identified to \mathcal{H}_* . Then ([C], equation (162)) we have $f_* = \Gamma \widetilde{f_{\text{III}}}$ where

$$\Gamma = \exp \left(\sum_{n \geq 2} (-1)^{n-1} \frac{f[y_n]}{n} y_1^n \right)$$

This implies that $f_*^{\text{inv}} f_* = \widetilde{f_{\text{III}}}^{\text{inv}} \Gamma^{\text{inv}} \Gamma \widetilde{f_{\text{III}}}$, and $\Gamma^{\text{inv}} \Gamma = \exp \left(\sum_{m \in \mathbb{N}^*} -2(f[y_{2m}]/2m) y_1^{2m} \right)$. Finally, for all $m \in \mathbb{N}^*$, we have $f[y_{2m}] \in \mathbb{Q}f[y_2]^m$. □

This proof implies also a relation between the two regularizations as follows :

Corollary 4.12. We have, where, for $n \in \mathbb{N}^*, \gamma_n \in \mathbb{Q}$:

$$((f_*^{\text{inv}} \circ \Sigma_{\omega, \text{Lie}}) f_*)[w] = (f_{\text{III}}^{-1} e_1 f_{\text{III}})[\Sigma_{\omega, \text{Lie}}(w)] + \sum_{\substack{u, u' \in \mathcal{H}_*, n \in \mathbb{N}^* \\ w = uy_{2n}u'}} \gamma_n f[y_2]^n (\widetilde{f_{\text{III}}}^{\text{inv}} \circ \Sigma_{\omega, \text{Lie}})[u] \widetilde{f_{\text{III}}}[u']$$

Proof. Indeed, since Γ involves only powers of y_1 , we have $\Gamma^{\text{inv}} \circ \Sigma_{\omega, \text{Lie}} = \Gamma^{\text{inv}}$. □

4.3.4. Other proof for the symmetry equation for images by Frobenius and analogues. Let Λ be a formal variable.

Let $s_d, \dots, s_1, t_{d'}, \dots, t_1 \in \mathbb{N}^*, i_1, \dots, i_{d'}, j_1, \dots, j_d \in \{1, \dots, r\}$, where $d, d' \in \mathbb{N}^*$.

Proof. The number

$$(-1)^{\sum_{i=1}^d s_i} (\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[\frac{1}{1-\Lambda e_0} e_z \frac{e_0^{s_1-1}}{(1-\Lambda e_0)^{s_1}} e_{z_{j_1}} \cdots \frac{e_0^{s_d-1}}{(1-\Lambda e_0)^{s_d}} e_{z_{j_d}} e_0^{t_{d'}-1} e_{z_{i_{d'}}} \cdots e_0^{t_1-1} e_{z_{i_1}} \right]$$

is equal, because of (8), to

$$(-1)^{\sum_{j=1}^{d'} t_j} (\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[e_{z_{i_1}} e_0^{t_1-1} \cdots e_{z_{i_{d'}}} e_0^{t_{d'}-1} e_{z_{j_d}} \frac{e_0^{s_d-1}}{(1+\Lambda e_0)^{s_d}} \cdots e_{z_{j_1}} \frac{e_0^{s_1-1}}{(1+\Lambda e_0)^{s_1}} e_z \frac{1}{1+\Lambda e_0} \right]$$

and then, because (9) to

$$(-1)^{\sum_{j=1}^{d'} t_j} (\Phi_{0z}^{-1} e_z \Phi_{0z}) \left[\frac{1}{1-\Lambda e_0} e_{z_{i_1}} \cdots \frac{e_0^{t_1-1}}{(1-\Lambda e_0)^{t_1}} e_{z_{i_{d'}}} \frac{e_0^{t_{d'}-1}}{(1-\Lambda e_0)^{t_{d'}}} e_{z_{j_d}} e_0^{s_d-1} \cdots e_{z_{j_1}} e_0^{s_1-1} e_z \right]$$

□

5. EQUATIONS OF ASSOCIATOR TYPE

5.1. Heuristics on algebraic relations and automorphisms of the fundamental group. What we want to understand is the correspondence between, on the one hand, relations satisfied by automorphisms of the fundamental group, such as the associator and Kashiwara-Vergne relations, and, on the other hand, the proof of algebraic relations between multiple harmonic sums, either via taking Taylor coefficients, or via taking infinite sums of algebraic relations.

Recall that the usual associator relations come from what follows : the automorphisms of the de Rham pro-unipotent fundamental groupoid induced by algebraic automorphisms of $M_{0,4}$ and $M_{0,5}$ are of finite order, since they consist of permutations of the marked points, which gives the desired relation. The Kashiwara-Vergne relations are, on the other hand, relations satisfied by a certain monodromy automorphism of the fundamental group.

We have chosen a reference tangential base-point O on $\overline{M_{0,n}}$ (and its images by the forgetful maps $\overline{M_{0,n}} \rightarrow \overline{M_{0,m}}$ which we will also denote by O), as the origin of the paths of integration (§3.1). We make the following observations :

i) Only the automorphisms of the pro-unipotent fundamental groupoid that induce an automorphism of $\pi_1^{un,dR}(M_{0,m}, O) \rightarrow \pi_1^{un,dR}(M_{0,m}, O)$ (with O in both sides) can lead to relations involving exclusively multiple harmonic sums by taking Taylor coefficients.

ii) The automorphisms of $M_{0,4}$ and $M_{0,5}$ inducing the associator relations can all be written in terms of automorphisms involved which do not fix O ; this is not true for the automorphisms which fix O . Indeed, the element of $\text{Aut}(\overline{M_{0,5}})$ inducing the pentagon relation does not fix O on $M_{0,5}$; the automorphism $z \mapsto 1-z$, giving the duality relation, does not fix 0 on \mathbb{P}^1 ; finally, the hexagon relation relies on $z \mapsto \frac{1}{z}$ and $z \mapsto \frac{z}{z-1}$. The last one fixes 0 , but can be express as a composition of $z \mapsto \frac{1}{z}$ and $z \mapsto 1-z$. Moreover, the only iterates of these automorphisms that fix O are the identity automorphisms.

iii) Consider the unique non-trivial automorphism of $M_{0,4}$ which fixes 0 , i.e. the homography $z \mapsto \frac{z}{z-1}$. Consider the relation that it induces, on the level of prime Taylor coefficients. We will explain in §6 that this relation is identical to the relation between the images by Frobenius of prime multiple harmonic sums and their analogues, provided by $e_0 + \Phi^{-1} e_1 \Phi + \Phi^{-1}(e_0, e_\infty) e_\infty \Phi(e_0, e_\infty) \equiv 0 \pmod{\zeta(2)}$.

The status of this equation is ambiguous. It is a part of Drinfeld's definition of GRT_1 , although Drinfeld instantly shows that it follows from the three other equations defining GRT_1 ([Dr], proposition 5.9). At the same time, it is in a natural way an equation concerning the automorphism μ_Φ , and a part of the Kashiwara-Vergne relations [AET]. We will view it here as a part of Kashiwara-Vergne equations.

iv) If we want to take infinite sums of algebraic relations to obtain relations satisfied by the images by Frobenius of prime multiple harmonic sums and their analogues, the Kashiwara-Vergne equations seem adapted, whereas the associator equation do not seem adapted - more precise statements will be given in §5 and §6. Basically, this is because the monodromy automorphism μ_Φ , subject to Kashiwara-Vergne relations, is naturally identified with an automorphism of $\pi_1^{\text{un},dR}(M_{0,4}, O)$ - or any chosen reference base point. The corresponding automorphism of $\mathcal{O}(\pi_1^{\text{un},dR}(M_{0,4}, \text{can}))$ satisfies a relation of almost commuting with the reindexation maps Σ_ω of §3.3 (we will give in §6 a precise version of this statement).

For all those reasons, we will state pre-associator relations in §5 for sequences of numbers $(\text{Li } \mathcal{T})_{\text{Orb } O}$, induced by automorphisms of $M_{0,4}$, $M_{0,5}$ that do not fix the reference base-point only.

As for the relations between multiple harmonic sums induced by algebraic automorphisms of $M_{0,n}$ fixing O , others than the one given by $(z \mapsto \frac{z}{z-1})_*$, we do not know yet whether and how they would be related to, for example, relations between the images by Frobenius of prime multiple harmonic sums given Kashiwara-Vergne relations.

5.2. Pre-associator equations and their relative analogues. These are the relations between multiple polylogarithms that induce the associator relations.

Definition 5.1. Let "pre-associator" equations be the equations of horizontality for ∇_{KZ} on the bundle of paths that start at O of the following automorphisms :

- i) the automorphism $(z \mapsto 1 - z)_*$ of $\pi_1^{\text{un},dR}(M_{0,4})$ (pre-duality equation)
- ii) the automorphism $(z \mapsto \frac{1}{z})_*$ of $\pi_1^{\text{un},dR}(M_{0,4})$ (with the previous one, they form a pre-hexagon equation)
- iii) the automorphism σ_* of $\pi_1^{\text{un},dR}(M_{0,5})$ where $\sigma : (x_1, x_2, x_3, x_4, x_5) \mapsto (x_5, x_4, x_1, x_3, x_2)$. In cubic coordinates : $(c_1, c_2) \mapsto (c_2, \frac{1-c_2}{1-c_1 c_2})$ (pre-pentagon equation)

These are now some relative analogues of those automorphisms.

Let the sets

$$i_{4,n} = \{f : \{1, 2, 3, 4\} \rightarrow \{1, \dots, n+3\} \mid f(2) = n+1, f(3) = n+2, f(4) = n+3\}$$

$$i_{5,n} = \{f : \{1, 2, 3, 4, 5\} \rightarrow \{1, \dots, n+3\} \mid f(3) = n+1, f(4) = n+2, f(5) = n+3\}$$

By attaching to $\{1, \dots, l\}$ the canonical coordinates (x_1, \dots, x_l) on $M_{0,l}$, an element of $i_{4,n}$, $i_{5,n}$ can be seen as a morphism $\overline{M_{0,4}} \rightarrow \overline{M_{0,n}}$, resp. $\overline{M_{0,5}} \rightarrow \overline{M_{0,n}}$, and also as a morphism of groups : $\text{Aut } \overline{M_{0,4}} \rightarrow \text{Aut } \overline{M_{0,n}}$, resp. $\text{Aut } \overline{M_{0,5}} \rightarrow \text{Aut } \overline{M_{0,n}}$.

Definition 5.2. Let "relative analogues of pre-associator equations" be the equations of horizontality for ∇_{KZ} on the bundle of paths that start at O of the images by $i_{4,n}$ and $i_{5,n}$ of the automorphisms $(z \mapsto 1 - z)_*^{rel}$; $(z \mapsto \frac{1}{z})_*^{rel}$; and σ_* .

Let us now give an idea of the shape of these automorphisms.

Proposition 5.3. For every $f \in i_{4,n}$, the image by f of the automorphism $(z \mapsto \frac{1}{z})_*$ resp. $(z \mapsto 1 - z)_*$ of $\overline{M_{0,4}}$ is the one given in simplicial coordinates by $(y_1, \dots, y_n) \mapsto (1 - y_1, \dots, 1 - y_n)$ and $(y_1, \dots, y_n) \mapsto (\frac{1}{y_1}, \dots, \frac{1}{y_n})$.

Proof. The automorphisms $(z \mapsto \frac{1}{z})_*$ and $(z \mapsto 1 - z)_*$ of $\overline{M_{0,4}}$ are given respectively as the transposition exchanging x_2 and x_4 , resp. x_2 and x_3 . Thus, the result follows from the definition of simplicial coordinates. \square

As for the 5-cycle σ , let us consider all the possible embeddings $M_{0,5} \rightarrow M_{0,n+3}$ of the form $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_i, x_j, x_{n+1}, x_{n+2}, x_{n+3})$.

Concretely speaking, take for example a curve $\mathbb{P}^1 - Z$ with Z stable by $z \mapsto 1 - z$, i.e. of the form $\{0, 1, z_1, \dots, z_r, 1 - z_1, \dots, 1 - z_r, \infty\}$. The KZ equation is :

$$\nabla_{\text{KZ}}^{\mathbb{P}^1 - Z} : f \mapsto df - \left(\sum_{i=1}^r \frac{dx}{x - z_i} e_{z_i} + \sum_{i=1}^r \frac{dx}{x - (1 - z_i)} e_{z_i} + \frac{dx}{x} e_0 + \frac{dx}{x - 1} e_1 \right) f$$

Let $L_Z^{(0)}$, resp. $L_Z^{(1)}$ be the unique solution on $\mathbb{P}^1 - Z$ which is holomorphic and such that $L(x) \exp(-e_0 \log(x))$ is a holomorphic function with value 1 at $x = 0$. By standard arguments, we have

$$L_Z^{(0)} = L_Z^{(1)} \Phi_{01}(Z)$$

for an element $\Phi_{01}(Z) \in \pi_1^{un}(\mathbb{P}^1 - Z, 0, 1)$ is a variant of the Knizhnik-Zamolodchikov associator, and

$$L_Z^{(1)}(e_{a_i}) = L_Z^{(0)}(1 - z)(e_{1-a_i})$$

The two equations can be combined into :

$$L_Z^{(0)}(z) = L_Z^{(0)}(1 - z)(e_{1-a_i}) \Phi_{01}(Z)$$

5.3. Taylor coefficients. As follows from our discussion of §5.1, we can call pre-associator equations the following ones :

Definition 5.4. i) Let $\text{Orb } O$ be the set of images of O by the elements of $i_{4,n}$ and $i_{5,n}$.
ii) For w an index, let $(\text{Li } \mathcal{T})_{\text{Orb } O}[w]$ be the sequence of Taylor coefficients attached to w of the chosen branch of multiple polylogarithms on $M_{0,n}$ at elements of $\text{Orb } O$.

5.4. Images by Frobenius and analogues.

Definition 5.5. Consider the set of images of $(\Phi_{0z}^{-1} e_z \Phi_{0z})^{\mathcal{M}} \left[\frac{1}{1 - \Lambda e_0} e_z w \right]$ by the embeddings $\pi_1^{un,dR}(X_Z) \hookrightarrow \pi_1^{un,dR}(M_{0,n})$ induced by the exact sequence of Lemma 2.13. (Let $(\text{Li } \mathcal{T})_{\text{Orb } O, \text{prime}}^{\mathcal{M}}$ be the set of their images by elements of $\text{Orb } O$).

5.5. Statement. Theorem 2 : relative analogues of pre-associator equations
• Taylor coefficients

We have counterparts of the three relative analogues of pre-associator equations on the sequences of numbers $(\text{Li } \mathcal{T})_{\text{OrbO}}[w]$ and $(\text{Li } \mathcal{T})_{\text{OrbO,prime}}^{\mathcal{M}}[w]$.

Proof. Obtained by expressing that these automorphisms are of finite order. \square

Remark 5.6. Take the cyclotomic case of $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)$, over $\mathbb{Q}[\xi]$ where ξ is a primitive N -root of unity. The Taylor coefficients of multiple polylogarithms at 1 and ξ^k , for all k , are related to each other through the action of the automorphism $(z \mapsto \xi z)_*$ of $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)$. The map $(z \mapsto \frac{1}{z})_*$ induces also an automorphisms of the variety. It relates the Taylor coefficients at 0 and ∞ , and at ξ^k and ξ^{-k} for all k .

We also have a distribution relation and, when $N = 2$ or $N = 4$, there are relations induced by all the additional automorphisms of $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)$. The additional automorphisms for $N = 1$ have already been considered in the general setting above.

6. EQUATIONS OF KASHIWARA-VERGNE TYPE

6.1. Statement. In this paragraph, we state the analogues of Kashiwara-Vergne equations. The statements are less explicit than the ones of the double shuffle relations. Making them fully explicit requires additional work which concerns intrinsically Kashiwara-Vergne equations, and is not itself related to multiple harmonic sums. For this reason we leave it to a separate paper. We will prove there, before all, that the equations give relations given by absolutely convergent p -adic series when we consider the images by Frobenius of prime multiple harmonic sums.

More precisely, let us take the notations of §2.3.4, and consider the equation

$$(25) \quad \text{Ad } \Phi(t_{12}, t_{23}) \circ \mu_{\Phi}^{12,3} \circ \mu_{\Phi}^{1,2} = \mu_{\Phi}^{1,23} \circ \mu_{\Phi}^{2,3}$$

By applying it to the elements x_1, x_2, x_3 of the free Lie algebra that is involved, this equation yields three equalities between grouplike series. Then, we aim to translate each of these three equalities on certain infinite sums of words on $\{x_1, x_2, x_3\}$, such that the relation obtained involves the numbers of the form $(\Phi^{-1}e_1\Phi)[\frac{1}{1-e_0}e_1w]$.

Let $a_1 = \Phi(x, -x - y)$, $a_2 = \Phi(y, -x - y)$. The right-hand side of (25), $\mu_{\Phi}^{1,23} \circ \mu_{\Phi}^{2,3}$, maps

$$\begin{aligned} x_1 &\mapsto (a_1 x a_1^{-1})(x_1, x_2 + x_3) \\ x_2 &\mapsto (a_1 x a_1^{-1}) \left(a_2(x_1, x_2 + x_3).x_2.a_2(x_1, x_2 + x_3)^{-1}, a_2(x_1, x_2 + x_3).x_3.a_2(x_1, x_2 + x_3)^{-1} \right) \\ x_3 &\mapsto (a_2 y a_2^{-1}) \left(a_2(x_1, x_2 + x_3).x_2.a_2(x_1, x_2 + x_3)^{-1}, a_2(x_1, x_2 + x_3).x_3.a_2(x_1, x_2 + x_3)^{-1} \right) \end{aligned}$$

For the left hand side of (25), we can write a similar formula for $\mu_{\Phi}^{12,3} \circ \mu_{\Phi}^{1,2}$; it is the action of $\text{Ad } \Phi(t_{12}, t_{23})$ which is much heavier to write explicitly. We leave it to a separate paper on Kashiwara-Vergne equations.

Because of this restriction of nonexplicitness, also, we cannot know yet the potential

link between these relations and the relations concerning certain algebraic automorphisms of $M_{0,5}$ that stabilize the chosen base-point.

There is also, as a particular case, the "equation of the special automorphism", which can be seen as the one dimensional part of equation (25). It has the advantage to be fully explicit, and it has an interesting connection with a property satisfied, not by a monodromy automorphism but by an algebraic automorphism of $M_{0,4}$.

Theorem 3 : equations of Kashiwara-Vergne type

- **the general equations - for images by Frobenius and analogues**

The Kashiwara-Vergne type equation of [AET], i.e. equation (25) yields three families of relations between the motivic version of the images by Frobenius of prime multiple harmonic sums, which also involve multiple zeta values and their analogues.

- **the equation of the special automorphism, one dimensional part of the general equation**

The following equalities are true for all words $w \in \ker \tilde{\partial}_{e_0}$.

for Taylor coefficients : The following is obtained by considering Taylor coefficients of relations between multiple polylogarithms :

$$(26) \quad (\text{Li } \mathcal{T})_{O,\text{prime}}(w(e_0 + e_1, -e_1)) = - \sum_{z \in \ker \tilde{\partial}_{e_0}} (-1)^{\text{depth}(z)} (\text{Li } \mathcal{T})_{O,\text{prime}}(zw)$$

for images by Frobenius and analogues. We have :

$$(27) \quad (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{M}}(w(e_0 + e_1, -e_1)) = - \sum_{z \in \ker \tilde{\partial}_{e_0}} (-1)^{\text{depth}(z)} (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{M}}(zw)$$

where zw is the concatenation of words.

Remark 6.1. Concretely, we see by the formula $\mu_{\Phi}^{1,23} \circ \mu_{\Phi}^{2,3}$ that the general equation can be expressed by generalizations of the harmonic Ihara action that we defined in part II of our work 'the Frobenius horizontal isomorphism of the pro-unipotent fundamental group of curves $\mathbb{P}^1 - Z$ '.

Let us add to this another result on Taylor coefficients that could be related to the general equation of the theorem.

Proposition 6.2. Let the automorphism σ_O of $\overline{M_{0,5}}$ given in cubic coordinates on $M_{0,5}$ by

$$(c_1, c_2) \mapsto \left(-c_1 \frac{1-c_2}{1-c_1}, -c_2 \frac{1-c_1}{1-c_2} \right)$$

The horizontality of σ_O with respect to ∇_{KZ} yields a family of relations between multiple harmonic sums.

Remark 6.3. For any curve $\mathbb{P}^1 - Z$ over a discrete valuation ring of unequal characteristic and absolutely non-ramified, (i.e. to which we can apply [D], §11), the properties of the Frobenius map F_* defined there imply :

$$e_0 + \sum_{z \in Z - \{0\}} (\Phi_{p,-k})_{0z}^{-1} e_z (\Phi_{p,-k})_{0z} = 0$$

Let us now discuss the related work :

Remark 6.4. Based on Hoffman's work using Newton series, Rosen has proved in [R] an "asymptotic duality theorem" on prime multiple harmonic sums $p^{\text{weight}} H_p$: for all indices w , we have, for all primes p ,

$$(28) \quad (p^{\text{weight}} H_p)(w(e_0 + e_1, -e_1)) = (p^{\text{weight}} H_p)(w + (w * (\frac{1}{1 + y_1})))$$

We explain here how it is actually another expression of the equation of the special automorphism above, despite the slightly different formulations and proofs. We also have, for $w = y_{s_d} \dots y_{s_1}$,

$$(29) \quad (p^{\text{weight}} H_p)(w(e_0 + e_1, -e_1)) = (p^{\text{weight}} H_p)(y_{s_d+1} - y_{s_d})(y_{s_d-1} \dots y_{s_1} * \frac{1}{1 + y_1})$$

Finally, for all w and for all $t \in \mathbb{N}$, we have :

$$(p^{\text{weight}} H_p)(y_{t+1}(w * \frac{1}{1 + y_1})) = - \sum_{\substack{d \geq 1 \\ x_d, \dots, x_1 \geq 1 \\ z = y_{x_d} \dots y_{t+x_1}}} (-1)^{\text{depth}(z)} (p^{\text{weight}} H_p)(z.w)$$

hence the equality between (27), (28), (29). We will prove this remark in the next paragraph. By the same proof, Rosen's formula actually works for any power of p .

Note that, however, those three formulae are quite different combinatorially. We will use precisely the formulation (27) in part II for lifts of congruences.

6.2. Proofs and explicit formulas : Taylor coefficients.

6.2.1. *Automorphisms of $\overline{M_{0,4}}$ and their relative versions.* Let the automorphism of $\mathbb{P}^1 - \{0, 1, \infty\}$

$$z \mapsto \frac{z}{z - 1}$$

It the unique homography which preserves 0 and exchanges 1 and ∞ . The statement will come from using the automorphism of the fundamental groupoid of $\mathbb{P}^1 - \{0, 1, \infty\}$ that it induces. In order to obtain a maximal statement, we will consider the following, with $a, c, d \in \overline{\mathbb{Q}}$:

$$\sigma_{a,c,d} : z \mapsto \frac{az}{cz + d}$$

Proof. Let $(\begin{smallmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{smallmatrix})$ be any index. Consider a neighbourhood of 0 in \mathbb{C} , and the straight paths from 0 to points of the neighbourhood. For z in this neighbourhood, write the equality :

$$(30) \quad \int_0^{\sigma_{a,c,d}(z)} \omega_0^{s_d-1} \omega_{z_{i_d}} \dots \omega_0^{s_1-1} \omega_{z_{i_1}} = \int_0^z (\sigma_{a,c,d})^* (\omega_0^{s_d-1} \omega_{z_{i_d}} \dots \omega_0^{s_1-1} \omega_{z_{i_1}})$$

The left-hand side of the equality is

$$\text{Li} \left(\begin{smallmatrix} z_{i_d}, \dots, z_{i_1} \\ s_d, \dots, s_1 \end{smallmatrix} \right) \left(\frac{az}{cz + d} \right)$$

Let us write its Taylor coefficients at 0. For $z \in \mathbb{C}$ on a neighbourhood of 0, we have, for all $n \in \mathbb{N}$:

$$\left(\frac{az}{cz+d}\right)^n = \left(\frac{az}{d}\right)^n \sum_{l \geq 0} \binom{-n}{l} \left(\frac{c}{d}z\right)^l$$

Thus,

$$\begin{aligned} & \text{Li}\left(\frac{z_{i_d}, \dots, z_{i_1}}{s_d, \dots, s_1}\right) \left(\frac{az}{cz+d}\right) \\ &= \sum_{0 < n_1 < \dots < n_d} \frac{(z_{i_2}/z_{i_1})^{n_1} \dots (z_{i_d}/z_{i_{d-1}})^{n_{d-1}} (1/z_{i_d})^{n_d}}{n_1^{s_1} \dots n_d^{s_d}} \left(\frac{az}{cz+d}\right)^{n_d} \\ &= \sum_{0 < n_1 < \dots < n_d < m} \frac{(z_2/z_1)^{n_1} \dots (z_d/z_{d-1})^{n_{d-1}} (a/cz_d)^{n_d}}{n_1^{s_1} \dots n_d^{s_d}} \left(\frac{c}{d}\right)^m \binom{m-1}{m-n_d} \end{aligned}$$

We will now write binomial coefficients in terms of multiple harmonic sums. For all $N \in \{1, \dots, m-1\}$, we have

$$\binom{m-1}{m-N} = \frac{(m-1)(m-2) \dots (m-(m-N))}{1 \times 2 \times \dots \times (m-N)} = \left(\frac{m}{1} - 1\right) \left(\frac{m}{2} - 1\right) \dots \left(\frac{m}{m-N} - 1\right)$$

Expanding the product gives the following expression, where the sum over r is finite :

$$= \sum_{r \geq 0} m^r (-1)^{m-n-r} H_{m-n}(\underbrace{1, \dots, 1}_r)$$

By a change of variable, we have, for all $r \in \mathbb{N}$:

$$H_{m-N}(\underbrace{1, \dots, 1}_r) = \sum_{N < j_r < \dots < j_1 < m} \frac{1}{(m-j_r) \dots (m-j_1)} (-1)^r$$

Now, take $m = p^k$ with p a prime number and $k \in \mathbb{N}^*$. In the previous expression, expand into p -adic series factors of the form $\frac{1}{1-x} = \sum_{l \geq 0} x^l$ with $|x|_p < 1$. It gives :

$$\binom{p^k-1}{p^k-N} = (-1)^{N-1} \sum_{r \geq 0} \sum_{l_1, \dots, l_r \geq 0} (p^k)^{r + \sum_{i=1}^r l_i} \sum_{N < j_r < \dots < j_1 < p^k} \frac{1}{j_r^{1+l_r} \dots j_1^{1+l_1}}$$

Now we can write the desired Taylor coefficient :

$$\begin{aligned} & (p^k)^{s_d + \dots + s_1} \text{Li}\left(\frac{z_{i_d}, \dots, z_{i_1}}{s_d, \dots, s_1}\right) \left(\frac{az}{cz+d}\right) [z^{p^k}] = - \sum_{r \geq 0} \sum_{l_1, \dots, l_r \geq 0} (p^k)^{\sum_{j=1}^d s_j + r + \sum_{i=1}^r l_i} \\ & \times \sum_{0 < n_1 < \dots < n_d < j_r < \dots < j_1 < p^k} \frac{(z_{i_2}/z_{i_1})^{n_1} \dots (z_{i_d}/z_{i_{d-1}})^{n_{d-1}} (-a/cz_{i_d})^{n_d}}{n_1^{s_1} \dots n_d^{s_d} j_r^{l_r+1} \dots j_1^{l_1+1}} \left(\frac{c}{d}\right)^{p^k} \\ (31) \quad & = - \sum_{r \geq 0} \sum_{l_1, \dots, l_r \geq 0} (p^k)^{\text{weight}} H_{p^k} \left(\frac{-ad}{c^2}, \overbrace{\frac{-a}{c}, \dots, \frac{-a}{c}}^r, z_{i_d}, \dots, z_{i_1} \right) \\ & \quad 1 + l_r, \dots, 1 + l_1, s_d, \dots, s_1 \end{aligned}$$

The right-hand side of the equality is expressed via the fact that, for all $\xi \in \mathbb{C}$, we have :

$$\sigma_{a,c,d}^* \frac{dz}{z-\xi} = \left[\frac{(a-\xi c)}{(a-\xi c)z + (-\xi d)} - \frac{c}{cz+d} \right] dz$$

In the case of $z \mapsto \frac{z}{z-1}$, i.e. of $\sigma_{1,1,-1}$, this gives :

$$(32) \quad \sigma_{1,1,-1}^* \frac{dz}{z} = \frac{dz}{z} - \frac{dz}{z-1}, \quad \sigma_{1,1,-1}^* \frac{dz}{z-1} = \frac{dz}{z-1}$$

The formulas (30), (31), (32) imply the Taylor coefficient part of the statement. \square

Proof. of the remark 6.4 of §6.1 : going back to the previous proof, we can also write (as in [R] with $k = 1$)

$$\binom{p^k - 1}{p^k - n_d} = \binom{p^k - 1}{n_d - 1} = \binom{p^k - 1}{p^k - (p^k - n_d + 1)}$$

and apply the previous computation with $N = p^k - n_d - 1$ instead of $N = n_d$. This gives an expression of the form $\sum_{0 < n < p^k} H_n(w') H_n(w'')$, which, by the series shuffle formula, can be written as $\sum_{0 < n < p^k} H_n(w * w')$. \square

6.2.2. Automorphisms of $\overline{M}_{0,5}$. Here, we give details on the formulas of the automorphism σ_O evoked in proposition 6.2 §6.1. We have :

Lemma 6.5. We have :

$$\begin{aligned} \sigma^* \left(\frac{dc_1}{c_1} \right) &= \frac{dc_2}{c_2 - 1} + \frac{dc_1}{c_1} - \frac{dc_1}{c_1 - 1} \quad ; \quad \sigma^* \left(\frac{dc_1}{c_1 - 1} \right) = -\frac{d(c_1 c_2)}{1 - c_1 c_2} + \frac{dc_1}{c_1 - 1} \\ \sigma^* \left(\frac{dc_2}{c_2} \right) &= \frac{dc_1}{c_1 - 1} + \frac{dc_2}{c_2} - \frac{dc_2}{c_2 - 1} \quad ; \quad \sigma^* \left(\frac{dc_2}{c_2 - 1} \right) = -\frac{d(c_1 c_2)}{1 - c_1 c_2} + \frac{dc_2}{c_2 - 1} \\ \sigma^* \left(\frac{d(c_1 c_2)}{1 - c_1 c_2} \right) &= \frac{d(c_1 c_2)}{1 - c_1 c_2} \end{aligned}$$

Proof. Trivial. \square

Lemma 6.6. Let a power series $S(c_1, c_2) = \sum_{(n,m) \in \mathbb{N}^2} \alpha_{n,m} c_1^n c_2^m \in \mathbb{Q}[c_1, c_2]$. We have

$$S(\sigma_O(c_1, c_2)) = \sum_{N, M \geq 0} c_1^N c_2^M \sum_{\substack{K, u, v, l_1, l_2 \geq 0 \\ K+u+l_1=N \\ K+v+l_2=M \\ k_1+k_2=K}} \alpha_{k_1+u, k_2+v} \binom{-k_1-u}{l_1} \binom{-k_2-v}{l_2}$$

Proof. We have :

$$\begin{aligned} S(c_1, c_2) &= \sum_{(n,m) \in \mathbb{N}^2} \alpha_{n,m} c_1^n c_2^m = \sum_{(n,m) \in \mathbb{N}^2} \alpha_{n,m} \left(-\frac{c_1(1-c_2)}{1-c_1} \right)^n \left(-\frac{c_2(1-c_1)}{1-c_2} \right)^m \\ &= \sum_{(n,m) \in \mathbb{N}^2} \alpha_{n,m} (-c_1)^n (-c_2)^m \times \\ &\quad \sum_{\substack{l_1, l_2 \geq 0 \\ 0 \leq k_1 \leq n \\ 0 \leq k_2 \leq m}} \binom{-n}{l_1} (-c_1)^{l_1} \binom{-m}{l_2} (-c_2)^{l_2} \binom{n}{k_1} (-c_2)^{k_1} \binom{m}{k_2} (-c_1)^{k_2} \end{aligned}$$

For (c_1, c_2) at a neighbourhood of O , the double series is absolutely convergent and we can invert the two sums.

$$= \sum_{k_1, k_2 \geq 0} (-c_1)^{k_2} (-c_2)^{k_1} \sum_{\substack{u, v \geq 0 \\ l_1, l_2 \geq 0}} \alpha_{k_1+u, k_2+v} (-c_1)^{k_1+u} (-c_2)^{k_2+v} \binom{-n}{l_1} \binom{-m}{l_2} (-c_1)^{l_1} (-c_2)^{l_2}$$

$$\begin{aligned}
&= \sum_{k_1, k_2 \geq 0} (c_1 c_2)^{k_1 + k_2} \sum_{\substack{u, v \geq 0 \\ l_1, l_2 \geq 0}} \alpha_{k_1 + u, k_2 + v} (-c_1)^{u + l_1} (-c_2)^{v + l_2} \binom{-k_1 - u}{l_1} \binom{-k_1 - v}{l_2} \\
&= \sum_{N, M \geq 0} c_1^N c_2^M \sum_{\substack{K, u, v, l_1, l_2 \geq 0 \\ K + u + l_1 = N \\ K + v + l_2 = M \\ k_1 + k_2 = K}} \alpha_{k_1 + u, k_2 + v} \binom{-k_1 - u}{l_1} \binom{-k_2 - v}{l_2}
\end{aligned}$$

□

6.3. Proofs : images by Frobenius and analogues. The details concerning the general equation are left to another paper. We have to translate on coefficients of the form $\frac{1}{1-e_0}e_1w$ the equality

$$e_0 + (\Phi^{-1}e_1\Phi)(e_0, e_1) + (\Phi^{-1}e_1\Phi)(e_0, e_\infty) \equiv 0 \pmod{\zeta(2)}$$

Thus, let us write first the formula for the dual of the map

$$\sigma_{0\infty} = (z \mapsto \frac{z}{z-1})_* : \pi_1^{un, dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can}) \rightarrow \pi_1^{un, dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can})$$

given on points by $f(e_0, e_1) \mapsto f(e_0, e_\infty)$. Since the formulae relating the associators to multiple zeta values involve a $(-1)^{\text{depth}}$ factor, we also consider $r \circ \sigma_{0\infty}^\vee \circ r$, where

$$\begin{aligned}
r : \mathcal{O}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can}) &\rightarrow \mathcal{O}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can}) \\
(e_0, e_1) &\mapsto (e_0, -e_1)
\end{aligned}$$

Lemma 6.7. We have the formulas :

$$\begin{aligned}
\sigma_{0\infty}^\vee : \mathcal{O}(\pi_1^{\text{dR}}(X; \infty, 0)) &\longrightarrow \mathcal{O}(\pi_1^{\text{dR}}(X; 1, 0)) \\
w(e_0, e_1) &\longmapsto w(e_0 - e_1, -e_1)
\end{aligned}$$

(33)

$$\begin{aligned}
r_1 \circ \sigma_{0\infty}^\vee \circ r_\infty : \mathcal{O}(\pi_1^{un, dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can})) &\longrightarrow \mathcal{O}(\pi_1^{un, dR}(\mathbb{P}^1 - \{0, 1, \infty\}, \text{can})) \\
w(e_0, e_1) &\longmapsto w(e_0 + e_1, -e_1)
\end{aligned}$$

Proof. Reindex $f(e_0, e_\infty) = \sum_{w \text{ word}} f[w]w(e_0, -e_0 - e_1)$ in terms of the words on e_0, e_1 . □

Now, we see how the image of a formal sum of words $\frac{1}{1-e_0}e_1w$ by the maps of the previous lemma can be expressed in the form $\frac{1}{1-e_0}e_1w'$.

Lemma 6.8. We have $(1 - \Lambda(e_0 + e_1))^{-1} = (1 - \Lambda e_0)^{-1} + (1 - \Lambda e_0)^{-1} \Lambda e_1 (1 - \Lambda(e_0 + e_1))^{-1}$, which implies :

$$(r_1 \circ \sigma_{0\infty}^\vee \circ r_\infty) \circ \Sigma_{\omega, \text{Lie}} = -\Sigma_{\omega, \text{Lie}}(-\text{id} + \circ(r_1 \circ \sigma_{0\infty}^\vee \circ r_\infty) \circ \Sigma_{\omega, \text{Lie}})$$

Proof. The first equation is obtained by writing $1 = (1 - \Lambda e_0 - \Lambda e_1)(1 - \Lambda e_0 - \Lambda e_1)^{-1} = (1 - \Lambda e_0)(1 - \Lambda e_0 - \Lambda e_1)^{-1} - \Lambda e_1(1 - \Lambda e_0 - \Lambda e_1)^{-1}$ and multiplying by $1 - \Lambda e_0$ on the left. It implies easily the second equation. □

The combination of the two lemmas implies the wanted part of the theorem.

7. MOTIVIC GALOIS ACTION AND PERIOD MAPS

7.1. Formula for a motivic Galois action.

7.1.1. *From the residues to the monodromy.* In order to write the motivic coaction on the prime multiple harmonic sums motive, let us replace $\Phi_{0z}^{-1}e_1\Phi_{0z}$ by its exponential version :

$$\Phi_{0z}^{-1}\exp(2i\pi e_z)\Phi_{0z}$$

Considering $\Phi_{0z}^{-1}\exp(2i\pi e_z)\Phi_{0z}$ unveils the composition of paths which is hidden in $\Phi^{-1}e_1\Phi$; indeed, whereas e_z is the residue of ∇_{KZ} at 1, $\exp(2i\pi e_z)$ is its monodromy ; and, in particular :

Proposition 7.1. $\Phi_{0z}^{-1}\exp(2i\pi e_z)\Phi_{0z}$ is the regularized integration of ∇_{KZ} along the path $\gamma_z^{sym} = \gamma_z \circ c_z \circ \gamma_z$ defined as the conjugation on a positively oriented simple loop around z , based at $-\vec{1}_z$, by the usual path γ_z from $\vec{1}_0$ to $-\vec{1}_z$.

We can retrieve $\Phi_{0z}^{-1}e_z\Phi_{0z}$ from $\Phi_{0z}^{-1}e^{2i\pi e_z}\Phi_{0z}$ by two different ways : either as

$$2i\pi\Phi^{-1}e_1\Phi = \log(\Phi_{0z}^{-1}e^{2i\pi e_z}\Phi_{0z})$$

or in the following way :

Proposition 7.2. We have :

$$(\Phi_{0z}^{-1}\exp(2i\pi e_z)\Phi_{0z})^{\mathcal{M}} \equiv (2i\pi)^{\mathcal{M}}(\Phi_{0z}^{-1}e_z\Phi_{0z})^{\mathcal{M}} \mod ((2i\pi)^2)^{\mathcal{M}}$$

7.1.2. *Formula for the motivic Galois coaction.* We are going now to state and prove the theorem 4, giving the motivic Galois coaction on the prime multiple harmonic sum motive, with, as intermediate object, the iterated integrals

$$\Phi_{0z}^{-1}e^{2i\pi e_1}\Phi_{0z}\left[\frac{1}{1-\Lambda e_0}w\right]$$

To this end, let us review a few properties of the motivic Galois coaction.

Usually, when motivic multiple zeta values are dealt with using the motivic Galois coaction, the set of paths that is involved is made of the straight path dch from 0 to 1 and its inverse.

Here, we are going to use an extended set of paths including as well a loop around one. We have to be careful that some of the standard properties of motivic multiple zeta values commonly used are no longer true in this setting : the property $I(a_0; a_1, \dots, a_n, a_{n+1}) = 0$ if $a_1 = \dots = a_n$ ([Br3] §2.4, I0) is not true if the path of integration is the loop around 1. Instead, we have the weaker statement :

Lemma 7.3. If $a_1 = \dots = a_n = 0$, then $\tilde{I}_{\gamma_z^{sym}}(a_0; a_1, \dots, a_n; a_{n+1}) = 0$

We need also to use the two following properties : the first one is

Proposition 7.4. Goncharov's coaction formula is valid for iterated integrals on the loop around 1.

To write the second property, let us recall the formula of composition of paths for iterated integrals. Let two composable paths $\gamma_1 \in P^{top}(X_Z, b, a)$, $\gamma_2 \in P^{top}(X, c, b)$, where P^{top} denotes the set of topological paths extended to tangential base-points of X_Z . We have

$$I_{\gamma_2 \circ \gamma_1}(c; a_n, \dots, a_1; a) = \sum_{k=0}^n I_{\gamma_2}(c; a_n, \dots, a_{k+1}; b) I_{\gamma_1}(b; a_k, \dots, a_1; a)$$

Proposition 7.5. Goncharov's coaction formula commutes to the composition of paths.

Proposition 7.4 and proposition 7.5 imply :

Proposition 7.6. The formula

$$(34) \quad \Delta \tilde{I}(a_{n+1}; a_n, \dots, a_1; a_0) \\ = \sum_{0=i_0 < i_1 < \dots < i_k < i_{k+1}=n} \tilde{I}(a_{n+1}; a_{i_k}, \dots, a_{i_1}; a_0) \otimes \prod_{l=0}^k \tilde{I}(a_{i_{l+1}}; a_{i_{l+1}-1} \dots a_{i_l+1}; a_{i_l})$$

is also true for iterated integrals on γ_z^{sym} .

To state theorem 4, replace Δ , which acts on motivic iterated integrals, by the variant acting on differential forms : identify $(u_{n+1}; u_n, \dots, u_1; u_0)$ with the element $e_{u_n} \dots e_{u_1}$ of $\mathcal{O}(\pi_1^{un, dR}(X_Z, u_{n+1}, u_0)) \simeq \mathcal{H}_{\mathfrak{m}}(e_Z)$ and let :

$$\Delta^{dR} : \mathcal{O}(\pi_1^{un, dR}(X_Z, a_{n+1}, a_0)) \rightarrow \mathcal{O}(\pi_1^{un, dR}(X_Z, a_{n+1}, a_0)) \otimes_{i < j} \mathcal{O}(\pi_1^{un, dR}(X_Z, a_j, a_i))$$

$$(35) \quad (a_{n+1}; a_n, \dots, a_1; a_0) \mapsto \\ \sum_{0=i_0 < i_1 < \dots < i_k < i_{k+1}=n} (a_{n+1}; a_{i_k}, \dots, a_{i_1}; a_0) \otimes \otimes_{l=0}^k (a_{i_{l+1}}; a_{i_{l+1}-1} \dots a_{i_l+1}; a_{i_l})$$

consider a variant of our map of reindexation of the differential forms

$$(\Sigma'_{\omega})_{\mathfrak{m}} : \begin{array}{l} \mathcal{H}_{\mathfrak{m}}(e_Z) \mapsto \widehat{\mathcal{H}_{\mathfrak{m}}(e_Z)} \\ w \mapsto \frac{1}{1-e_0} w \end{array}$$

Consider now the tensor algebra $T(\mathcal{H}_{\mathfrak{m}}(e_Z)) = \oplus_{n \geq 0} \mathcal{H}_{\mathfrak{m}}(e_Z)^{\otimes n}$, and denote by

$$(\Sigma'_{\omega})_{\mathfrak{m}}^{\otimes} = \oplus_{n \geq 0} ((\Sigma'_{\omega})_{\mathfrak{m}} \otimes \text{id}^{\otimes n-1}) : T(\mathcal{H}_{\mathfrak{m}}(e_Z)) \rightarrow \oplus_{n \geq 0} \mathcal{H}_{\mathfrak{m}}(e_Z)^{\otimes n-1} \otimes \widehat{\mathcal{H}_{\mathfrak{m}}(e_Z)}$$

i.e. the map which applies $(\Sigma'_{\omega})_{\mathfrak{m}}^*$ only to the first tensor component. We will denote also $\mathfrak{m} : T(\mathcal{H}_{\mathfrak{m}}(e_Z)) \rightarrow \oplus_{n \geq 0} \mathcal{H}_{\mathfrak{m}}(e_Z)^{\otimes n-1}$ the shuffling of all tensor components.

Finally, denote by $\tilde{I}_{\gamma_z^{sym}}$ the Hodge-Tate structures arising from [G] lifting the iterated integrals on γ_z^{sym} .

Theorem 4. The coaction on $(\Phi_{0z}^{-1} e^{2i\pi e_z} \Phi_{0z}) \left[\frac{1}{1-\Lambda e_0} w \right]$ is given by

$$\tilde{I}_{\gamma_z^{sym}} \circ \Delta^{dR} \circ (\Sigma'_{\omega})_{\mathfrak{m}} = \tilde{I}_{\gamma_z^{sym}} \circ \mathfrak{m} \circ (\Sigma'_{\omega})_{\mathfrak{m}}^{\otimes} \circ \Delta^{dR}$$

Proof. We compute $\Delta^{dR} \circ (\Sigma'_{\omega})_{\mathfrak{m}}$ by injecting the formula defining $(\Sigma'_{\omega})_{\mathfrak{m}}$ into the formula (35) for Δ . We see that, by Lemma 7.3, in the formula (35), the factor $l = k$ on the right $I(a_{n+1}; \dots, a_{i_k})$ must contain at least one $a_i \neq 0$ in order to be non zero. Then, a simple reindexation of $\Delta^{dR} \circ (\Sigma'_{\omega})_{\mathfrak{m}}$ gives that it is equal to $\mathfrak{m} \circ (\Sigma'_{\omega})_{\mathfrak{m}}^{\otimes} \circ \Delta^{dR}$. Then, by proposition 7.6, we can apply Goncharov's formula to $\tilde{I}_{\gamma_z^{sym}}$. This gives the result. \square

Remark 7.7. In part III, we will take the dual point of view of the Ihara action. We will write a formula given on the level of the Ihara action, more directly adapted to $\Phi_{0z}^{-1} e_z \Phi_{0z}$.

7.2. Period maps and conjectures of periods. The algebras of motives and periods that we consider here are not just \mathbb{Q} -algebras as usual. They are complete topological algebras, with respect to the weight adic topology. Thus, it makes sense to define period maps as conjectural isomorphisms between topological complete \mathbb{Q} -algebras.

Let the weight-adic completion of the algebra of motivic hyperlogarithms $\widehat{\mathcal{Z}_{\mathcal{M}}}$; let $\widehat{\mathcal{Z}}$, resp. $\widehat{\mathcal{Z}}_p$, the analogous completion, in the sense of equation (22) of §3.3.1.d, i.e. : its elements are :

$$\sum_{(w_n)} \text{Li}[w] \Lambda^{\text{weight}(w)}$$

where (w_n) is a sequence of $\mathcal{H}_{\text{m}}(e_Z)$, such that each term is weight-homogeneous and $\text{weight}(w_n) \rightarrow \infty$.

Let us consider the complete sub algebras, $\mathcal{H}_{\text{pr.}}^{\mathcal{M}}$, $\mathcal{H}_{\text{pr.}}$, $\mathcal{H}_{\text{pr.}}^p$ generated by, respectively, the prime multiple harmonic sum motive, its formal complex and p -adic periods.

We start with the usual period map $\mathcal{Z}_{\mathcal{M}} \rightarrow \mathcal{Z}$, and its p -adic analogue $\mathcal{Z} \rightarrow \mathcal{Z}_p$, which has been constructed by Yamashita in an unpublished work. Since we can pre-composing these maps with the action of the weight : $\lambda \mapsto \text{multiplication by } \lambda^{\text{weight}}$, we have well-defined variants :

$$(36) \quad \mathcal{Z}_{\mathcal{M}} \rightarrow \mathcal{Z}[\Lambda] \quad \mathcal{Z}_{\mathcal{M}} \rightarrow \mathcal{Z}_p[\Lambda]$$

mapping $\text{Li}^{\mathcal{M}}(w) \mapsto \Lambda^{\text{weight}(w)} \text{Li}(w)$, resp. $\text{Li}^{\mathcal{M}}(w) \mapsto \Lambda^{\text{weight}(w)} \text{Li}_p(w)$.

Definition 7.8. Consider the restriction to the complete topological algebras topologically generated by the prime multiple harmonic sum motives, of the weight-adic completion of the period maps (36)

$$\begin{aligned} \text{per} : \quad & \mathcal{H}_{\text{pr.}}^{\mathcal{M}} \rightarrow \mathcal{H}_{\text{pr.}} \\ & (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{M}} \mapsto (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{Z}/\zeta(2)\mathcal{Z}[[\Lambda]]} \\ \text{per}_p : \quad & \mathcal{H}_{\text{pr.}}^{\mathcal{M}} \rightarrow \mathcal{H}_{\text{pr.}}^p \\ & (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{M}} \mapsto (\text{Li } \mathcal{T})_{O,\text{prime}}^{\mathcal{Z}_p[[\Lambda]]} \end{aligned}$$

These are the formal complex and p -adic period maps of the prime multiple harmonic sum motive.

The corresponding conjecture of periods is :

Conjecture 7.9. The maps per and per_p are isomorphisms of topological \mathbb{Q} -algebras equipped with the weight-adic topology.

We delay to part II the definition of the Taylor period map and its period conjecture.

8. REMARKS

8.1. Lifts to the framework where $(2i\pi)^{\mathcal{M}} \neq 0$. In the paragraph §7, we have used the monodromy $\Phi_{0z}^{-1} e^{2i\pi e_z} \Phi_{0z}$ and its coefficients of the form $(\Phi_{0z}^{-1} e^{2i\pi e_z} \Phi_{0z})[\frac{1}{1-\Lambda e_0} w]$, to obtain a formula for the motivic Galois coaction on the prime multiple harmonic sum

motive. This shows both the importance of the setting where $(2i\pi)^{\mathcal{M}} \neq 0$ for studying the prime multiple harmonic sum motive, and the natural role of the monodromy in the study.

It suggests the following question : are there lifts of the algebraic relations of theorems 1 and 3 in the algebra $\widehat{\mathcal{Z}_{\mathcal{M}}}$ non modded out by $(2i\pi)^{\mathcal{M}}$, satisfied by, either, the natural lift of the prime multiple harmonic sum motive - the one defined by the same formula i.e. $(\Phi_{0z}^{-1}e_z\Phi_{0z})[\frac{1}{1-e_0}w]$ - or the variant $(\Phi_{0z}^{-1}e^{2i\pi e_z}\Phi_{0z})[\frac{1}{1-\Lambda e_0}w]$?

Here are some answers to this question :

Proposition 8.1. i) The integral shuffle relation of theorem 1 remains true for the natural lift of the prime multiple harmonic sum motive in \mathcal{Z}_M .

ii) The series shuffle relation of theorem 1 does not remain true. Instead, it is true by replacing, in $(\Phi_{0z}^{-1}e_z\Phi_{0z})[\frac{1}{1-e_0}w]$, Φ_{0z} and Φ_{0z}^{-1} by their series-shuffle regularized version, i.e. respectively $\Gamma_z\Phi_{0z}$ and $\Phi_{0z}^{-1}\Gamma_z^{-1}$ with Γ_z an point of $\pi_1^{un,dR}(X_Z, -\vec{1}_z, -\vec{1}_z)$.

This follows directly from the proof of theorem 1.

The analogue of the integral shuffle formula of theorem 1 for the coefficients $(\Phi_{0z}^{-1}e^{2i\pi e_z}\Phi_{0z})[\frac{1}{1-\Lambda e_0}w]$ seems to be of different nature : it seems to us that the natural point of view, here, is to use Lemma 4.6, and write, with several formal variables Λ, Λ' and to apply to it $\Phi^{-1}e^{2i\pi e_1}\Phi$, which satisfies the usual shuffle equation - by contrast with $\Phi^{-1}e_1\Phi$ which satisfies the shuffle equation modulo products. This gives instantly.

Proposition 8.2. We have, for all $w, w' \in \mathcal{H}_*(Z)$, and for all $i, j \in \{1, \dots, r\}$:

$$(37) \quad (\Phi^{-1}e^{2i\pi e_1}\Phi)[\Lambda\Sigma_{\omega, \text{Lie}}(z_i)\Lambda^{-1}(w)] \times (\Phi^{-1}e^{2i\pi e_1}\Phi)[\Lambda'\Sigma_{\omega, \text{Lie}}(z_j)\Lambda'^{-1}(w')] \\ = (\Phi^{-1}e^{2i\pi e_1}\Phi)[(\Lambda + \Lambda')\Sigma_{\omega, \text{Lie}}(z_i)(\Lambda + \Lambda')^{-1}\left(w \boxplus (\Lambda')\Sigma_{\omega, \text{Lie}}(z_j)(\Lambda')^{-1}(w')\right) \\ + (\Phi^{-1}e^{2i\pi e_1}\Phi)[(\Lambda + \Lambda')\Sigma_{\omega, \text{Lie}}(z_j)(\Lambda + \Lambda')^{-1}\left((\Lambda\Sigma_{\omega, \text{Lie}}(z_i)\Lambda^{-1})(w) \boxplus w'\right)]$$

Remark 8.3. When we specify complex or p -adic values of Λ and Λ' , we obtain an equality on values $(\Phi^{-1}e^{2i\pi e_1}\Phi)[\frac{1}{1-\xi e_0}e_{z_i}w]$ for different values ξ at the same time. We think that these must be seen as a "functional equation" on the map $\xi \mapsto (\Phi^{-1}e^{2i\pi e_1}\Phi)[\frac{1}{1-\xi e_0}e_{z_i}w]$. The reference to functional equations comes from that, the maps $\xi \mapsto \Phi[\frac{1}{1-\xi e_0}e_1w]$ are nothing else than multiple polygamma functions, iterated series analogues of the classical polygamma functions.

Remark 8.4. If we apply instead $\Phi^{-1}e^{2i\pi e_1}\Phi$ to proposition 4.8, as in the proof of theorem 1, we obtain a formula mixing the coefficients $(\Phi^{-1}e^{2i\pi e_1}\Phi)[\frac{1}{1-\Lambda e_0}w]$ and usual coefficients $(\Phi^{-1}e^{2i\pi e_1}\Phi)[w]$. This is again because $\Phi^{-1}e^{2i\pi e_1}\Phi$ satisfies the usual shuffle equation.

In the case of the Kashiwara-Vergne relations, the theory contains already a solution to the problem : it is standard that

Proposition 8.5. While $\Phi^{-1}e_1\Phi$ satisfies the Lie version of the Kashiwara-Vergne equations, $\Phi^{-1}e^{2i\pi e_1}\Phi$ satisfies the group version.

The Kashiwara-Vergne equations for $\Phi^{-1}e^{2i\pi e_1}\Phi$ are then the natural lift of the Kashiwara-Vergne equations for $\Phi^{-1}e_1\Phi$, which are re-obtained as their coefficient of degree one with respect to $2i\pi$.

For example, the natural lift of the equation

$$e_0 + (\Phi^{-1}e_1\Phi)(e_0, e_1) + (\Phi^{-1}e_1)\Phi(e_0, e_\infty) = 0 \pmod{\zeta(2)}$$

is

$$\exp(2i\pi e_0) \cdot \Phi^{-1} \exp(2i\pi e_1) \Phi \cdot \Phi^{-1}(e_0, e_\infty) \exp(2i\pi e_\infty) \Phi(e_0, e_\infty) = 1$$

and so on for the rest of Kashiwara-Vergne relations.

We do not know yet whether these equation can be translated nicely on the coefficients of the form $(\Phi^{-1}e^{2i\pi e_1}\Phi)[(\Lambda\Sigma_{\omega, \text{Lie}}(z_i)\Lambda^{-1})w]$.

8.2. The shifting of the bounds of prime multiple harmonic sums. The map of reindexation of differential forms $(\Sigma_\omega)_*$, introduced in 3.3, has a particular interpretation in the context of prime multiple harmonic sums.

We take the usual notations : $Z = \{0, z_1, \dots, z_r, 1, \infty\}$ a finite subset of $\overline{\mathbb{Q}}$, with $z_0 = 0$ and $z_{r+1} = 1$; $d \in \mathbb{N}^*$, $(s_d, \dots, s_1) \in (\mathbb{N}^*)^d$, $i_{d+1}, \dots, i_1 \in \{1, \dots, r\}$. We have :

$$(38) \quad \Lambda(\Sigma_\omega)_* \Lambda^{-1} \left(\begin{array}{c} 1, z_{i_d}, \dots, z_1 \\ s_d, \dots, s_1 \end{array} \right) = \sum_{l_1, \dots, l_d} (-T)^{l_1 + \dots + l_d} \prod_{i=1}^d \binom{-s_i}{l_i} \left(\begin{array}{c} 1, z_{i_d}, \dots, z_1 \\ s_d + l_d, \dots, s_1 + l_1 \end{array} \right) \in \widehat{\mathcal{H}_*(e_Z)}$$

Fact 8.6. If $\lambda \in \mathbb{Z}$, the value at λ of $\Lambda(\Sigma_\omega)_* \Lambda^{-1}$

$$(39) \quad \Lambda(\lambda)(\Sigma_\omega)_* \Lambda(\lambda)^{-1} (\text{Li } \mathcal{T})_{O, \text{prime}} \left(\begin{array}{c} 1, z_{i_d}, \dots, z_1 \\ s_d, \dots, s_1 \end{array} \right) = \left((p^k)^{s_d + \dots + s_1} \sum_{\lambda p^k < n_1 < \dots < n_r < (\lambda+1)p^k} \frac{\left(\frac{z_{i_2}}{z_{i_1}}\right)^{n_1} \dots \left(\frac{1}{z_{i_d}}\right)^{n_d}}{n_1^{s_1} \dots n_d^{s_d}} \right) \in \prod_{k \in \mathbb{N}^*} \prod_p \overline{\mathbb{Q}}_p$$

A priori, this leads to define a "shifted multiple harmonic sum motive" ; by the compatibility of $(\Sigma_\omega)_*$ with the usual algebraic operations, already studied in this paper, the algebraic relations can a priori be adapted easily ; the shifted multiple harmonic sum motive admits as a particular period the terms of the last equality above.

This shows that the class of finite sums that we can understand is larger than we could expect at first.

8.3. Relations arising from other automorphisms of $M_{0,n}$. Let us recall that all relations among multiple zeta values are conjecturally implied by relating expressed in terms of $M_{0,4}$ and $M_{0,5}$. A priori, in this context, a similar conjecture can be made, using instead X_Z and $X_Z^2 - \Delta_Z$.

The double shuffle relations in higher dimensions can be decomposed into the double

shuffle relations in dimension 1 and 2.

On the associator and Kashiwara-Vergne side, remember that $\text{Aut}(\overline{M_{0,n+3}}) = S_{n+3}$ by a theorem of Mella and Bruno ; a priori, the symmetric group on $n+3$ element is generated by the 5-cycles that we have considered. As a consequence, we should obtain no new relations by considering more general automorphisms of $\overline{M_{0,n+3}}$.

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